Nonlinear turbulent magnetic diffusion and mean-field dynamo

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The nonlinear coefficients defining the mean electromotive force (i.e., the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity, the nonlinear κ tensor, etc.) are calculated for an anisotropic turbulence. A particular case of an anisotropic background turbulence (i.e., the turbulence with zero-mean magnetic field) with one preferential direction is considered. It is shown that the toroidal and poloidal magnetic fields have different nonlinear turbulent magnetic diffusion coefficients. It is demonstrated that even for a homogeneous turbulence there is a nonlinear effective velocity that exhibits diamagnetic or paramagnetic properties depending on the anisotropy of turbulence and the level of magnetic fluctuations in the background turbulence. The diamagnetic velocity results in the field being pushed out from the regions with stronger mean magnetic field, while the paramagnetic velocity causes the magnetic field to be concentrated in the regions with stronger field. Analysis shows that an anisotropy of turbulence strongly affects the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity, and the nonlinear α effect. Two types of nonlinearities (algebraic and dynamic) are also discussed. The algebraic nonlinearity implies a nonlinear dependence of the mean electromotive force on the mean magnetic field. The dynamic nonlinearity is determined by a differential equation for the magnetic part of the α effect. It is shown that for the $\alpha\Omega$ axisymmetric dynamo the algebraic nonlinearity alone (which includes the nonlinear α effect, the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity, etc.) cannot saturate the dynamo generated mean magnetic field while the combined effect of the algebraic and dynamic nonlinearities limits the mean magnetic field growth.

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I. INTRODUCTION

Generation of magnetic fields by the turbulent flow of a conducting fluid is a fundamental problem that has a large number of applications in solar physics, astrophysics, geophysics, planetary physics, etc. In recent time the problem of nonlinear mean-field magnetic dynamo is a subject of active discussions (see, e.g., Refs. [1–10]). It was suggested in Ref. [11] that the quenching of the nonlinear α effect is very strong and it causes a very weak saturated mean magnetic field. However, the later suggestion is in disagreement with observations of galactic and solar magnetic fields (see, e.g., Refs. [12–17]) and with numerical simulations (see, e.g., Refs. [18–20]).

Saturation of the dynamo generated mean magnetic field is caused by the nonlinear effects, i.e., by the back reaction of the mean magnetic field on the α effect, turbulent magnetic diffusion, differential rotation, etc. The evolution of the mean magnetic field $\mathbf{B}$ is determined by equation

$$\partial \mathbf{B}/\partial t = \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}+\boldsymbol{\mathcal{E}}-\eta\boldsymbol{\nabla}\times\mathbf{B}), \tag{1}$$

where $\mathbf{V}$ is a mean velocity (e.g., the differential rotation) and η is the magnetic diffusion due to the electrical conductivity of fluid. The mean electromotive force $\boldsymbol{\mathcal{E}}=\langle \mathbf{u}\times\mathbf{b}\rangle$ in an anisotropic turbulence is given by

$$\mathcal{E}_i=\alpha_{ij}B_j+(\mathbf{V}^{\rm eff}\times\mathbf{B})_i-\eta_{ij}(\boldsymbol{\nabla}\times\mathbf{B})_j-\kappa_{ijk}(\partial\hat{B})_{jk} -[\boldsymbol{\delta}\times(\boldsymbol{\nabla}\times\mathbf{B})]_i \tag{2}$$

(see Refs. [21,22]), where $(\partial\hat{B})_{ij}=(1/2)(\nabla_i B_j+\nabla_j B_i)$; $\mathbf{u}$ and $\mathbf{b}$ are fluctuations of the velocity and magnetic field, respectively; angular brackets denote averaging over an ensemble of turbulent fluctuations; the tensors α_{ij} and η_{ij} describe the α effect and turbulent magnetic diffusion, respectively; $\mathbf{V}^{\rm eff}$ is the effective diamagnetic (or paramagnetic) velocity; κ_{ijk} and $\boldsymbol{\delta}$ describe a nontrivial behavior of the mean magnetic field in an anisotropic turbulence. Nonlinearities in the mean-field dynamo imply dependencies of the coefficients $(\alpha_{ij},\eta_{ij},\mathbf{V}^{\rm eff}$, etc.) defining the mean electromotive force on the mean magnetic field. The α effect and the differential rotation are the sources of the generation of the mean magnetic field, while the turbulent magnetic diffusion and the κ effect (which is determined by the tensor κ_{ijk}) contribute to the dissipation of the mean magnetic field.

In spite of the nonlinear α effect, being under active study (see, e.g. [5,7]), the nonlinear turbulent magnetic diffusion, the nonlinear κ-effect, the nonlinear diamagnetic, and paramagnetic effects, etc. are poorly understood.

In the present paper we derived equations for the nonlinear turbulent magnetic diffusion, the nonlinear effective velocity, the nonlinear κ effect, etc. for an anisotropic turbulence. The obtained results for the nonlinear mean electromotive force are specified for an anisotropic background turbulence with one preferential direction. The background turbulence is the turbulence with zero mean magnetic field. We demonstrated that toroidal and poloidal magnetic fields have different nonlinear turbulent magnetic diffusion coefficients. It is shown that even for a homogeneous turbulence there is a nonlinear effective velocity that can be a diamagnetic or paramagnetic depending on the anisotropy of turbulence and the level of magnetic fluctuations in the background turbulence.

II. THE GOVERNING EQUATIONS

In order to derive equations for the nonlinear turbulent magnetic diffusion and other nonlinear coefficients defining the mean electromotive force, we will use a mean-field approach in which the magnetic $\mathbf{H}$ and velocity $\mathbf{v}$ fields are divided into the mean and fluctuating parts: $\mathbf{H}=\mathbf{B}+\mathbf{b}$, $\mathbf{v}=\mathbf{V}+\mathbf{u}$, where the fluctuating parts have zero mean values, $\mathbf{V}=\langle\mathbf{v}\rangle=\mathrm{const}$, and $\mathbf{B}=\langle\mathbf{H}\rangle$. The momentum equation and the induction equation for the turbulent fields $\mathbf{u}$ and $\mathbf{b}$ in a frame moving with a local velocity of the large-scale flows $\mathbf{V}$ are given by

$$\partial\mathbf{u}/\partial t=-\boldsymbol{\nabla}P'/\rho-[\mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{b})]/(\mu_0\rho)+\mathbf{T}+\nu\Delta\mathbf{u}+\mathbf{F}/\rho, \tag{3}$$

$$\partial\mathbf{b}/\partial t=\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}-\eta\boldsymbol{\nabla}\times\mathbf{b})+\mathbf{G}, \tag{4}$$

and $\boldsymbol{\nabla}\cdot\mathbf{u}=0$, where P' are the fluctuations of the hydrodynamic pressure, $\mathbf{F}$ is a random external stirring force, ν is the kinematic viscosity, η is the magnetic diffusion due to the electrical conductivity of fluid, ρ is the density of fluid, μ_0 is the magnetic permeability of the fluid, the nonlinear terms $\mathbf{T}$ and $\mathbf{G}$ are given by $\mathbf{T}=\langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\rangle-(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}+[\langle\mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{b})\rangle-\mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{b})]/(\mu_0\rho)$, and $\mathbf{G}=\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{b}-\langle\mathbf{u}\times\mathbf{b}\rangle)$. We consider the case of large hydrodynamic ($\mathrm{Re}=l_0u_0/\nu\gg1$) and magnetic ($\mathrm{Rm}=l_0u_0/\eta\gg1$) Reynolds numbers, where u_0 is the characteristic velocity in the maximum scale l_0 of turbulent motions.

A. The procedure of the derivation of equation for the nonlinear mean electromotive force

The procedure of the derivation of equation for the nonlinear mean electromotive force is as follows (for details, see Appendix A).

(a) By means of Eqs. (3) and (4) we derive equations for the second moments:

$$f_{ij}(\mathbf{k},\mathbf{R})=\int\langle u_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}=f_{ji}(-\mathbf{k},\mathbf{R}), \tag{5}$$

$$h_{ij}(\mathbf{k},\mathbf{R})=\int\langle b_i(\mathbf{k}+\mathbf{K}/2)b_j(-\mathbf{k}+\mathbf{K}/2)\rangle\times\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}/\mu_0\rho=h_{ji}(-\mathbf{k},\mathbf{R}), \tag{6}$$

$$g_{ij}(\mathbf{k},\mathbf{R})=\int\langle b_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}, \tag{7}$$

where $\mathbf{R}$ and $\mathbf{K}$ correspond to the large scales, and $\mathbf{r}$ and $\mathbf{k}$ to the small ones, i.e., $\mathbf{R}=(\mathbf{x}+\mathbf{y})/2$, $\mathbf{r}=\mathbf{x}-\mathbf{y}$, $\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2$, $\mathbf{k}=(\mathbf{k}_1-\mathbf{k}_2)/2$.

(b) We split all correlation functions (i.e., f_{ij},h_{ij},g_{ij}) into two parts, e.g., $h_{ij}=h_{ij}^{(N)}+h_{ij}^{(S)}$, where the tensor $h_{ij}^{(N)}=[h_{ij}(\mathbf{k},\mathbf{R})+h_{ij}(-\mathbf{k},\mathbf{R})]/2$ describes the nonhelical part of the tensor and $h_{ij}^{(S)}=[h_{ij}(\mathbf{k},\mathbf{R})-h_{ij}(-\mathbf{k},\mathbf{R})]/2$ determines the helical part of the tensor. Such splitting is caused, e.g., by different times of evolution of the helical and nonhelical parts of the magnetic tensor. In particular, the characteristic time of evolution of the tensor $h_{ij}^{(N)}$ is of the order $\tau_0=l_0/u_0$, while the relaxation time of the tensor $h_{ij}^{(S)}$ is of the order of $\tau_0\mathrm{Rm}$ (see, e.g., [15,23–25]).

(c) Equations for the second moments contain higher moments and a problem of closing the equations for the higher moments arises. Various approximate methods have been proposed for the solution of problems of this type (see, e.g. [26–28]). The simplest procedure is the τ approximation, which is widely used in the theory of kinetic equations. For magnetohydrodynamic turbulence this approximation was used in Ref. [29] (see also [7,30,31]). In the simplest variant, it allows us to express the third moments in terms of the second moments

$$M_{ij}-M_{ij}^{(0)}=-(f_{ij}-f_{ij}^{(0)})/\tau(k), \tag{8}$$

$$R_{ij}^{(N)}=-(h_{ij}^{(N)}-h_{ij}^{(0N)})/\tau(k), \tag{9}$$

$$C_{ij}=-g_{ij}/\tau(k), \tag{10}$$

where M_{ij}, R_{ij}, and C_{ij} are the third moments in equations for f_{ij},h_{ij}, and g_{ij}, respectively [see Eqs. (A3)–(A5) in Appendix A]. The superscript (0) corresponds to the background magnetohydrodynamic turbulence (it is a turbulence with zero mean magnetic field, $\mathbf{B}=\mathbf{0}$), $h_{ij}^{(0N)}$ is the nonhelical part of the tensor of magnetic fluctuations of the background turbulence, and $\tau(k)$ is the characteristic relaxation time of the statistical moments. We applied the τ approximation only for the nonhelical part $h_{ij}^{(N)}$ of the tensor of magnetic fluctuations because the corresponding helical part $h_{ij}^{(S)}$ is determined by an evolutionary equation (see, e.g., [15,23,32,24,2,25,8] and Sec. III C). Here we took into account magnetic fluctuations that can be generated by a stretch-twist-fold mechanism when a mean magnetic field is zero (see, e.g., [33,34]). This implies that $h_{ij}^{(0)}\neq0$. In inertia range of background turbulence $R_{ij}(\mathbf{B}=\mathbf{0})=0$ and $C_{ij}(\mathbf{B}=\mathbf{0})=0$. We also took into account that the cross-helicity tensor g_{ij} for $\mathbf{B}=\mathbf{0}$ is zero, i.e., $g_{ij}(\mathbf{B}=\mathbf{0})=0$.

The τ approximation is in general similar to eddy damped quasinormal Markovian (EDQNM) approximation. However, some principal difference exists between these two approaches (see [26,28]). The EDQNM closures do not relax to equilibrium, and this procedure does not describe properly the motions in the equilibrium state in contrast to the τ approximation. Within the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached [26]. In the τ approximation, the relaxation time for small departures from equilibrium is determined by the random motions in the equilibrium state, but not by the departure from equilibrium [26]. We use the τ approximation, but not the EDQNM approximation because we consider a case with $l_0|\boldsymbol{\nabla}B^2|/\mu_0\ll\langle\rho u^2\rangle$. As follows from the analysis of Ref. [26], the τ

approximation describes the relaxation to equilibrium state (the background turbulence) much more accurately than the EDQNM approach.

In this study we consider an intermediate nonlinearity that implies that the mean magnetic field is not strong enough in order to affect the correlation time of turbulent velocity field. The theory for a very strong mean magnetic field can be corrected after taking into account a dependence of the correlation time of the turbulent velocity field on the mean magnetic field.

(d) We assume that the characteristic time of variation of the mean magnetic field $\mathbf{B}$ is substantially larger than the correlation time $\tau(k)$ for all turbulence scales. This allows us to get a stationary solution for the equations for the second moments f_{ij}, h_{ij}, and g_{ij}. Using these equations [see Eqs. (A14)–(A21) in Appendix A] we calculate the electromotive force $\mathcal{E}_i(\mathbf{r}=0)=\int \mathcal{E}_i(\mathbf{k})d\mathbf{k}$, where $\mathcal{E}_i(\mathbf{k})=(1/2)\varepsilon_{imn}[g_{nm}^{(N)}(\mathbf{k},\mathbf{R})-g_{mn}^{(N)}(-\mathbf{k},\mathbf{R})]$. The result is given by

$$\mathcal{E}_i(\mathbf{r}=0)=a_{ij}B_j+b_{ijk}B_{j,k}, \tag{11}$$

where $B_{i,j}=\partial B_i/\partial R_j$,

$$a_{ij}=i\int \tau(1+\psi)^{-1}\varepsilon_{imn}k_j(f_{nm}^{(0S)}-h_{nm}^{(S)})d\mathbf{k}, \tag{12}$$

$$b_{ijk}=\int \tau(1+\psi)^{-1}[\varepsilon_{ijn}(f_{kn}^{(0N)}+h_{kn}^{(0N)})-2\varepsilon_{imn}k_{mj}h_{nk}^{(N)}]d\mathbf{k} \tag{13}$$

(for details, see Appendix A), $k_{ij}=k_ik_j/k^2$, $h_{nm}^{(N)}=h_{nm}^{(0N)}+\psi(1+2\psi)^{-1}(f_{nm}^{(0N)}-h_{nm}^{(0N)})$, ε_{ijk} is the Levi-Civita tensor, and $\psi=[(\boldsymbol{\beta}\cdot\mathbf{k})u_0\tau/2]^2$, $\beta_i=4B_i/(u_0\sqrt{2\mu_0\rho})$, $f_{ij}^{(0N)}$ and $f_{ij}^{(0S)}$ describe the nonhelical and helical tensors of the background turbulence.

(e) Following Ref. [21] we use an identity $B_{j,k}=(\partial\hat{B})_{jk}-\varepsilon_{jkl}(\boldsymbol{\nabla}\times\mathbf{B})_l/2$, which allows us to rewrite Eq. (11) for the electromotive force in the form

$$\mathcal{E}_i=\alpha_{ij}B_j+(\mathbf{U}\times\mathbf{B})_i-\eta_{ij}(\boldsymbol{\nabla}\times\mathbf{B})_j-\kappa_{ijk}(\partial\hat{B})_{jk}, \tag{14}$$

where

$$\alpha_{ij}(\mathbf{B})=(a_{ij}+a_{ji})/2, \quad U_k(\mathbf{B})=\varepsilon_{kji}a_{ij}/2, \tag{15}$$

$$\eta_{ij}=(\varepsilon_{ikp}b_{jkp}+\varepsilon_{jkp}b_{ikp})/4, \quad \kappa_{ijk}(\mathbf{B})=-(b_{ijk}+b_{ikj})/2. \tag{16}$$

B. The model for the background turbulence

For the integration in $\mathbf{k}$ space in Eqs. (12) and (13) we have to specify a model for the background turbulence (i.e., turbulence with zero mean magnetic field). We assume that the background turbulence is anisotropic and incompressible. The second moments for the turbulent velocity and magnetic fields of the background turbulence are given by

$$\tau c_{ij}(\mathbf{k})=(5/4)\{P_{ij}(k)[(2/5)\tilde{\eta}_T^{(a)}(\mathbf{k})-\mu_{mn}^{(a)}(\mathbf{k})k_{nm}] + 2[\delta_{ij}\mu_{mn}^{(a)}(\mathbf{k})k_{nm}+\mu_{ij}^{(a)}(\mathbf{k})-\mu_{im}^{(a)}(\mathbf{k})k_{mj} - k_{im}\mu_{mj}^{(a)}(\mathbf{k})]\} \tag{17}$$

(see Ref. [7]), where $c_{ij}=f_{ij}^{(0N)}$ when $a=v$, and $c_{ij}=h_{ij}^{(0N)}$ when $a=h$, and $\tilde{\eta}_T^{(v)}(\mathbf{k})=\tau f_{pp}^{(0N)}(\mathbf{k})$, $\tilde{\eta}_T^{(h)}(\mathbf{k})=\tau h_{pp}^{(0N)}(\mathbf{k})$, $P_{ij}(k)=\delta_{ij}-k_{ij}$, δ_{mn} is the Kronecker tensor. The anisotropic part of this tensor $\mu_{mn}^{(a)}(\mathbf{k})$ has the properties: $\mu_{mn}^{(a)}(\mathbf{k})=\mu_{nm}^{(a)}(\mathbf{k})$ and $\mu_{pp}^{(a)}(\mathbf{k})=0$. Inhomogeneity of the background turbulence is assumed to be weak, i.e., in Eq. (17) we dropped terms $\sim O(\boldsymbol{\nabla}(\eta_T^{(a)};\mu_{ij}^{(a)}))$, where $\eta_T^{(v)}=\tau_0u_0^2/3$, $\eta_T^{(h)}=\tau_0b_0^2/3\mu_0\rho$, and b_0 is the characteristic value of the magnetic fluctuations in the background turbulence. To integrate over k in Eqs. (12) and (13) we use the Kolmogorov spectrum of the background turbulence, i.e., $\tau f_{pp}^{(0N)}(\mathbf{k})=\eta_T^{(v)}\varphi(k)$, $\tau h_{pp}^{(0N)}(\mathbf{k})=\eta_T^{(h)}\varphi(k)$ and $\mu_{mn}^{(a)}(\mathbf{k})=\mu_{mn}^{(a)}(\mathbf{R})\varphi(k)/3$, where $\varphi(k)=(\pi k^2k_0)^{-1}(k/k_0)^{-7/3}$, $\tau(k)=2\tau_0(k/k_0)^{-2/3}$, and $k_0=l_0^{-1}$. We take into account that the inertial range of the turbulence exists in the scales: $l_d\leqslant r\leqslant l_0$. Here the maximum scale of the turbulence $l_0\ll L_B$, and $l_d=l_0/\mathrm{Re}^{3/4}$ is the viscous scale of turbulence, and L_B is the characteristic scale of variations of the nonuniform mean magnetic field.

In the following section we present results for the nonlinear coefficients defining the mean electromotive force.

III. NONLINEAR COEFFICIENTS DEFINING THE MEAN ELECTROMOTIVE FORCE

The procedure described in Sec. II (see also, for details Appendix A) allows us to calculate the nonlinear turbulent magnetic diffusion tensor, the nonlinear $\boldsymbol{\kappa}$ tensor, the nonlinear $\boldsymbol{\alpha}$ tensor, and the nonlinear effective drift velocity.

A. Nonlinear turbulent magnetic diffusion tensor and nonlinear κ tensor

The general form of the turbulent magnetic diffusion tensor $\eta_{ij}(\mathbf{B})$ contains all possible tensors: δ_{ij}, $\mu_{ij}^{(a)}$ β_{ij} and their symmetric combination $\overline{\mu_{ij}^{(a)}}=\mu_{in}^{(a)}\beta_{nj}+\beta_{in}\mu_{nj}^{(a)}$ [see Eq. (A51) in Appendix A], where $\beta_{ij}=\beta_i\beta_j/\beta^2$, and $\beta_i=4B_i/(u_0\sqrt{2\mu_0\rho})$. For an isotropic background turbulence (when $\mu_{ij}^{(a)}=0$) the turbulent magnetic diffusion tensor $\eta_{ij}(\mathbf{B})$ is given by

$$\eta_{ij}(\mathbf{B})=\delta_{ij}\{A_1(\sqrt{2}\beta)\eta_T^{(v)}+[A_1(\beta)-A_1(\sqrt{2}\beta)]\eta_T^{(h)}\} + (1/2)\beta_{ij}A_2(\beta)(\eta_T^{(v)}+\eta_T^{(h)}), \tag{18}$$

where the functions $A_k(\beta)$ are defined in Appendix B. For $\beta\ll 1$ Eq. (18) reads

$$\eta_{ij}(\mathbf{B})=\delta_{ij}[\eta_T^{(v)}-(2\beta^2/5)(2\eta_T^{(v)}-\eta_T^{(h)})] -(2/5)\beta_i\beta_j(\eta_T^{(v)}+\eta_T^{(h)}), \tag{19}$$

and for $\beta\gg 1$ it is given by

$$\eta_{ij}(\mathbf{B})=(3\pi/10\beta)\{\sqrt{2}\delta_{ij}[\eta_T^{(v)}+\eta_T^{(h)}(\sqrt{2}-1)] - \beta_{ij}(\eta_T^{(v)}+\eta_T^{(h)})\}. \tag{20}$$

The mean magnetic field causes an anisotropy of the turbulent magnetic diffusion tensor that is determined by the tensor β_{ij}. Magnetic fluctuations of the background turbulence contribute to the turbulent magnetic diffusion tensor $\eta_{ij}(\mathbf{B})$ in the nonlinear case. It follows from Eq. (20) that for $\beta \gg 1$ the tensor $\eta_{ij} \propto 1/\beta$.

The κ tensor describes a nontrivial behavior of the mean magnetic field in an anisotropic turbulence. For an isotropic background turbulence the κ tensor vanishes in spite of an anisotropy caused by the mean magnetic field. For an anisotropic background turbulence a general form of the κ tensor is given by Eq. (A52) in Appendix A. For $\beta \ll 1$ this tensor is given by

$$\kappa_{ijk}=-(1/6)(3\hat{L}_{ijk}^{(v)}+\hat{L}_{ijk}^{(h)})+(1/7)\beta^2(5\hat{L}_{ijk}^{(v)}+\hat{L}_{ijk}^{(h)}-4\hat{N}_{ijk}^{(v)} + 2\hat{N}_{ijk}^{(h)}) \tag{21}$$

and for $\beta \gg 1$ it reads

$$\kappa_{ijk}=-(\pi/16\beta)(\sqrt{2}-1)[\hat{L}_{ijk}^{(v)}+\hat{L}_{ijk}^{(h)}+3(\hat{N}_{ijk}^{(v)}+\hat{N}_{ijk}^{(h)})], \tag{22}$$

where $\hat{L}_{ijk}^{(a)}=\varepsilon_{ijn}\mu_{nk}^{(a)}+\varepsilon_{ikn}\mu_{nj}^{(a)}$ and $\hat{N}_{ijk}^{(a)}=\mu_{np}^{(a)}(\varepsilon_{ijn}\beta_{pk}+\varepsilon_{ikn}\beta_{pj})$. Note that for $\beta \gg 1$ the tensor $\kappa_{ijk} \propto 1/\beta$. The κ tensor contributes to the turbulent magnetic diffusion of the toroidal and poloidal mean magnetic fields (see Sec. V).

B. The hydrodynamic part of the nonlinear α tensor

Using Eqs. (12) and (15) we get

$$\alpha_{ij}^{(v)}(\mathbf{B},\mathbf{R})=\int \frac{\alpha_{ij}^{(v)}(0,\mathbf{k},\mathbf{R})}{1+\psi(\mathbf{B},\mathbf{k})}d\mathbf{k}, \tag{23}$$

where hereafter $\alpha_{ij}^{(v)}(0,\mathbf{k},\mathbf{R}) \equiv \alpha_{ij}^{(v)}(\mathbf{B}=0,\mathbf{k},\mathbf{R})$. Analysis in Refs. [7,22] shows that a form of the tensor $\alpha_{ij}^{(v)}(0,\mathbf{k},\mathbf{R})$ in an anisotropic turbulence can be constructed using the tensors k_{ij}, k_{ijmn} and ν_{ij}, where $k_{ijmn}=k_ik_jk_mk_n/k^4$ and ν_{ij} is the anisotropic part of the hydrodynamic contribution of the α tensor. Thus we use the following model for the tensor $\alpha_{ij}^{(v)}(0,\mathbf{k},\mathbf{R})$

$$\alpha_{ij}^{(v)}(0,\mathbf{k},\mathbf{R})=\{2\alpha_0^{(v)}(\mathbf{R})k_{ij}+(1-\epsilon)[\nu_{ip}(\mathbf{R})k_{pj} + \nu_{jp}(\mathbf{R})k_{pi}]+5\epsilon k_{ijmn}\nu_{mn}(\mathbf{R})\}\varphi(k)/2, \tag{24}$$

where the parameter ϵ describes an anisotropy of the helical component of turbulence and it changes in the interval: $0 \leq \epsilon \leq 1$. Here $\alpha_{ij}^{(v)}(\mathbf{0},\mathbf{R})=\int \alpha_{ij}^{(v)}(\mathbf{0},\mathbf{k},\mathbf{R})d\mathbf{k}=\alpha_0^{(v)}(\mathbf{R})\delta_{ij}+\nu_{ij}(\mathbf{R})$, and $\alpha_0^{(v)}(\mathbf{R})=(1/3)\alpha_{pp}^{(v)}(\mathbf{0},\mathbf{R})$, the anisotropic part $\nu_{ij}(\mathbf{R})$ of the hydrodynamic contribution of the α tensor has the properties $\nu_{ij}=\nu_{ji}$ and $\nu_{pp}=0$. Substituting Eq. (24) into Eq. (23), and using identities (A30) and (A31) we obtain the nonlinear dependence of the hydrodynamic part of the α effect on mean magnetic field

$$\alpha_{ij}^{(v)}(\mathbf{B})=(1/2)\{\delta_{ij}[2\{A_1(\beta)+A_2(\beta)\}\alpha_0^{(v)}+(1-\epsilon)A_2(\beta)\nu_\beta+5\epsilon\{C_2(\beta)+3C_3(\beta)\}\nu_\beta]+\nu_{ij}[(1-\epsilon)\{2A_1(\beta)+A_2(\beta)\}+10\epsilon\{C_1(\beta)+C_2(\beta)\}]\}, \tag{25}$$

where $\nu_\beta(\mathbf{R})=\nu_{mn}(\mathbf{R})\beta_{nm}$ and the functions $C_k(\beta)$ are defined in Appendix B. For $\epsilon=0$ Eq. (25) coincides with that derived in Ref. [7]. The asymptotic formulas for $\alpha_{ij}^{(v)}$ for $\beta \ll 1$ and $\beta \gg 1$ are given by Eqs. (A54) and (A57) in Appendix A.

C. The mean electromotive force and the nonlinear magnetic α tensor

Using Eqs. (A41), (A44), and (A47) we calculate the electromotive force $\boldsymbol{\mathcal{E}}$

$$\mathcal{E}_i=\alpha_{ij}B_j+(\mathbf{V}^{\mathrm{eff}}\times\mathbf{B})_i-\eta_{ij}(\boldsymbol{\nabla}\times\mathbf{B})_j-\kappa_{ijk}(\partial\hat{B})_{jk}, \tag{26}$$

where the nonlinear effective drift velocity $\mathbf{V}^{\mathrm{eff}}=\mathbf{U}+\mathbf{V}^{(N)}$ and the velocity $U_i(\mathbf{B})=-(1/2)\varepsilon_{imn}a_{mn}=-(1/2)\nabla_p\Lambda_{pi}^{(M)}(\sqrt{2}\beta)$ (see Ref. [7]), the velocity $\mathbf{V}^{(N)}$ is given by Eq. (A45), and the tensor of turbulent magnetic diffusion η_{ij} is given by Eq. (A51), the tensor κ_{ijk} is determined by Eq. (A52), and the tensor $\Lambda_{ij}^{(M)}$ is defined in Eqs. (A26) and (A34). In the kinematic dynamo the effective drift velocity (turbulent diamagnetic velocity) is caused by an inhomogeneity of turbulence. The effective drift velocity $\mathbf{U}(\mathbf{B})$ is determined by the tensor a_{ij} and is due to an induced inhomogeneity of turbulence caused by the nonuniform mean magnetic field. This implies that the nonuniform mean magnetic field modifies turbulent velocity field and creates the inhomogeneity of turbulence. The effective velocity $\mathbf{V}^{(N)}(\mathbf{B})$ is determined by tensor b_{ijk} and is caused by the nonuniform mean magnetic field.

The α tensor is determined by the hydrodynamic and magnetic contributions, i.e., $\alpha_{ij}(\mathbf{B})=\alpha_{ij}^{(v)}(\mathbf{B})+\alpha_{ij}^{(h)}(\mathbf{B})$ with

$$\alpha_{ij}^{(h)}(\mathbf{B})=\alpha_0^{(h)}(\mathbf{B})\Phi(\beta)\delta_{ij} \tag{27}$$

(see Ref. [7]), where the tensor $\alpha_{ij}^{(v)}(\mathbf{B})$ is determined by Eq. (25), the function $\Phi(\beta)=(3/\beta^2)[1-\arctan(\beta)/\beta]$, and the magnetic part $\alpha_0^{(h)}(\mathbf{B})$ of the α effect is determined by the dynamic equation

$$\frac{\partial\alpha_0^{(h)}}{\partial t}+\frac{\alpha_0^{(h)}}{T}+\boldsymbol{\nabla}\cdot(\mathbf{W}\alpha_0^{(h)}+\mathbf{F}_{\mathrm{flux}})=-\frac{4}{9\eta_T\mu_0\rho}\boldsymbol{\mathcal{E}}(\mathbf{B})\cdot\mathbf{B} \tag{28}$$

(see Refs. [8,25,23,24,35]), where $W_i=\tilde{c}_{ij}V_j$ is the velocity that depends on the mean fluid velocity $\mathbf{V}$ (for an isotropic turbulence the tensor $\tilde{c}_{ij}=\delta_{ij}$ and for an anisotropic turbulence with one preferential direction, say in the direction **e**, the tensor $\tilde{c}_{ij} = (23/30)\delta_{ij} + (7/10)e_i e_j$, see Ref. [25]); the flux

$$\mathbf{F}_{\rm flux} \propto \tau \alpha^{(v)}(\mathbf{B}) \frac{\nabla \rho}{\rho} \left(\frac{\eta_T^{(v)}(\mathbf{B}) B^2}{\eta_T^{(v)}(\mathbf{B}=0)\mu_0 \rho} \right) \tag{29}$$

is related to the flux of the magnetic helicity and is independent of the mean fluid velocity **V** [8] (see also Ref. [10]), and $T \sim \tau_0 \mathrm{Rm}$ is the characteristic time of relaxation of magnetic helicity. The asymptotic formulas for $\alpha_{ij}^{(h)}$ for $\beta \ll 1$ and $\beta \gg 1$ are given by Eqs. (A55) and (A58) in Appendix A.

IV. ANISOTROPIC BACKGROUND TURBULENCE WITH ONE PREFERENTIAL DIRECTION

Now we consider an anisotropic background turbulence with one preferential direction, say along an unit vector **e**. Thus the tensor $\eta_{ij}^{(v)}(\mathbf{B}=0) = \langle \tau v_i v_j \rangle$ is given by $\eta_{ij}^{(v)}(\mathbf{B}=\mathbf{0}) = \eta_T^{(v)}\delta_{ij} + \mu_{ij}^{(v)} = \eta_0^{(v)}\delta_{ij} + \varepsilon_\mu^{(v)} e_{ij}$, where the trace $\eta_{pp}^{(v)}(\mathbf{B}=0)$ in this equation yields $\eta_0^{(v)} = \eta_T^{(v)} - (1/3)\varepsilon_\mu^{(v)}$ and $e_{ij} = e_i e_j$. Therefore, the anisotropic part $\mu_{ij}^{(v)}$ of the tensor $\eta_{ij}^{(v)}(\mathbf{B}=\mathbf{0})$ is given by $\mu_{ij}^{(v)} = \varepsilon_\mu^{(v)}[e_{ij} - (1/3)\delta_{ij}]$ and $\overline{\mu_{ij}^{(v)}} \equiv \mu_{in}^{(v)}\beta_{nj} + \beta_{in}\mu_{nj}^{(v)} = \varepsilon_\mu^{(v)}[(e_i\beta_j + e_j\beta_i)(\mathbf{e}\cdot\hat{\boldsymbol{\beta}}) - (2/3)\beta_{ij}]$, where $\varepsilon_\mu^{(v)}$ is a degree of an anisotropy of the turbulence, and $\mu_\beta^{(v)} \equiv (1/2)\overline{\mu_{pp}^{(v)}} = \varepsilon_\mu^{(v)}[(\mathbf{e}\cdot\hat{\boldsymbol{\beta}})^2 - 1/3]$. It follows from these equations that $\eta_{ij}^{(v)}(\mathbf{B}=\mathbf{0}) = \delta_{ij}[\eta_T^{(v)} - (1/3)\varepsilon_\mu^{(v)}] + e_{ij}\varepsilon_\mu^{(v)}$. Now we take into account that the components $\eta_{xx}^{(v)}(\mathbf{B}=\mathbf{0})$, $\eta_{yy}^{(v)}(\mathbf{B}=\mathbf{0})$, and $\eta_{zz}^{(v)}(\mathbf{B}=\mathbf{0})$ are positive. This yields $-3/2 \leq \varepsilon_\mu^{(v)}/\eta_T^{(v)} \leq 3$. The equations for the corresponding magnetic tensors are obtained from these equations after the change $v \to h$. For the magnetic fluctuations we also obtain that $-3/2 \leq \varepsilon_\mu^{(h)}/\eta_T^{(h)} \leq 3$.

For galaxies, e.g., the preferential direction **e** is along rotation (which is parallel to the effective gravity field). For the axisymmetric $\alpha\Omega$ dynamo and large magnetic Reynolds numbers the toroidal magnetic field is much larger than the poloidal field. Therefore, the value $\mathbf{e}\cdot\hat{\boldsymbol{\beta}}$ is very small and can be neglected because $\hat{\boldsymbol{\beta}}$ is approximately directed along the toroidal magnetic field.

Thus, the nonlinear coefficients defining the mean electromotive force in a turbulence with one preferential direction are given by

$$\eta_{ij}(\mathbf{B}) = M_\eta \delta_{ij} + M_e e_{ij} + M_\beta \beta_{ij}, \tag{30}$$

$$\mathbf{V}^{\rm eff}(\mathbf{B}) = (1/B^2)[M_V^{(1)} \nabla B^2 + M_V^{(2)} \mathbf{e}(\mathbf{e}\cdot\nabla)B^2], \tag{31}$$

$$\kappa_{ijk}(\mathbf{B})(\partial \hat{B})_{jk} = M_\kappa [\mathbf{e} \times (\mathbf{e}\cdot\nabla)\mathbf{B}]_i \tag{32}$$

(see Appendix C), where we assumed that $\mathbf{e}\cdot\hat{\boldsymbol{\beta}} = \mathbf{0}$, the functions M_η, M_e, M_β, M_κ, $M_V^{(1)}$, and $M_V^{(2)}$ are given by Eqs. (C4)–(C9) in Appendix C. The tensor $\eta_{ij}(\mathbf{B})$ contains three tensors δ_{ij}, e_{ij}, and β_{ij} since here there are two preferred directions, along the vectors **e** and **B**.

Now we consider the hydrodynamic part of the α effect for an anisotropic background turbulence with one preferential direction. The tensor $\alpha_{ij}^{(v)}(\mathbf{B}=\mathbf{0})$ in this case can be rewritten in the form $\alpha_{ij}^{(v)}(\mathbf{B}=\mathbf{0}) = \alpha_0^{(v)}\delta_{ij} + \nu_{ij} = [\alpha_0^{(v)} - (1/3)\varepsilon_\alpha]\delta_{ij} + \varepsilon_\alpha e_{ij}$, where ε_α is a degree of an anisotropy of the α-tensor. Thus, the anisotropic part ν_{ij} is given by $\nu_{ij} = \varepsilon_\alpha[e_{ij} - (1/3)\delta_{ij}]$. The electromotive force contains the tensor α_{ij} in the form $\alpha_{ij}B_j$. Thus, $\nu_{ij}\hat{\beta}_j = -(1/3)\varepsilon_\alpha[\hat{\beta}_i - 3(\mathbf{e}\cdot\hat{\boldsymbol{\beta}})e_i]$ and $\nu_\beta = -(1/3)\varepsilon_\alpha[1 - 3(\mathbf{e}\cdot\hat{\boldsymbol{\beta}})^2]$. Using Eq. (25) we obtain the hydrodynamic part of the α tensor in an anisotropic background turbulence with one preferential direction

$$\begin{aligned} \alpha_{ij}^{(v)}(\mathbf{B}) = \delta_{ij}\{&[A_1(\beta) + A_2(\beta)][\alpha_0^{(v)} - (1/3)\varepsilon_\alpha(1-\epsilon)] \\ &- (5/6)\varepsilon_\alpha \epsilon [2C_1(\beta) + 3C_2(\beta) + 3C_3(\beta)]\} \\ \equiv{}& \Phi_\alpha(\beta)\delta_{ij}, \end{aligned} \tag{33}$$

where we assumed that $\mathbf{e}\cdot\hat{\boldsymbol{\beta}} = \mathbf{0}$. For $\epsilon \neq 0$ the tensor $\alpha_{ij}^{(v)}(\mathbf{B})$ can change its sign at some value B_* of the mean magnetic field [see Eqs. (C17) and (C21) in Appendix C]. Thus the point $B = B_*$ can determine a steady state configuration of the mean magnetic field for $\epsilon \neq 0$.

V. APPLICATIONS: MEAN-FIELD EQUATIONS FOR THE THIN-DISK AXISYMMETRIC $\alpha\Omega$ DYNAMO

Here we apply the obtained results for the nonlinear mean electromotive force to the analysis of the thin-disk axisymmetric $\alpha\Omega$ dynamo. Using Eqs. (30)–(32) we derive the mean-field equations for the thin-disk axisymmetric $\alpha\Omega$ dynamo,

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial z}\left(\eta_B \frac{\partial B}{\partial z}\right) + \tilde{G}\frac{\partial A}{\partial z}, \tag{34}$$

$$\frac{\partial A}{\partial t} = \eta_A \frac{\partial^2 A}{\partial z^2} - V_A \frac{\partial A}{\partial z} + \alpha B, \tag{35}$$

where r, φ and z are cylindrical coordinates, $\mathbf{B} = B\mathbf{e}_\varphi + \nabla \times (A\mathbf{e}_\varphi)$, $\tilde{G} = -r(\partial\Omega/\partial r)$, and

$$\eta_A(\mathbf{B}) = M_\eta + M_\kappa + M_\beta, \quad \eta_B(\mathbf{B}) = M_\eta + M_\kappa - 2M_V, \tag{36}$$

$$V_A(\mathbf{B}) = (\eta_A - \eta_B)(\ln|B|)',$$

$$\alpha(\mathbf{B}) = \Phi_\alpha(\mathbf{B})\alpha_0^{(v)} + \Phi(\mathbf{B})\alpha_0^{(h)}(\mathbf{B}), \tag{37}$$

and $F' = \partial F/\partial z$, $M_V = M_V^{(1)} + M_V^{(2)}$. In the axisymmetric problem $\partial \mathbf{B}/\partial\varphi = \mathbf{0}$. The thin-disk approximation implies that the spatial derivatives of the mean magnetic field with respect to z are much larger than the derivatives with respect to r. It is seen from Eqs. (30)–(32) and Eqs. (36) and (37) that the contributions to the turbulent diffusion coefficients $\eta_A(B)$ and $\eta_B(B)$ are from the tensor of turbulent diffusion $\eta_{ij}(\mathbf{B})$, the tensor $\kappa_{ijk}(\mathbf{B})$ and the nonlinear velocity $\mathbf{U}(\mathbf{B}) + \mathbf{V}^{(N)}(\mathbf{B})$. On the other hand, contributions to the effective

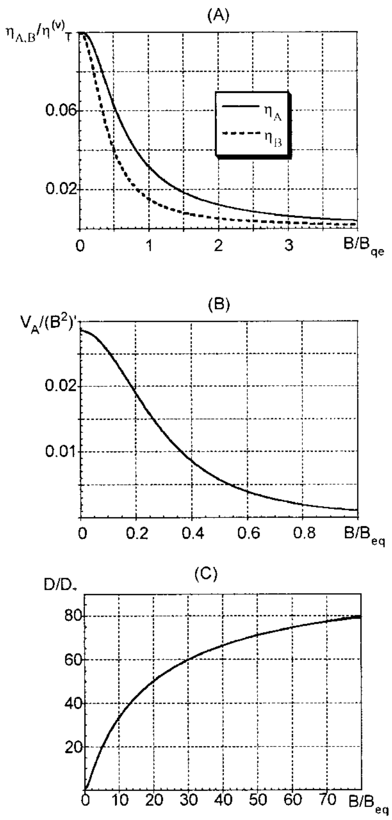

FIG. 1. (a) Nonlinear turbulent magnetic diffusion coefficients; (b) the nonlinear effective velocity; (c) the nonlinear dynamo number for $\eta_T^{(h)}=0$, $\varepsilon_\mu^{(v)}=-1.35\eta_T^{(v)}$, and $\varepsilon_\mu^{(h)}=0$.

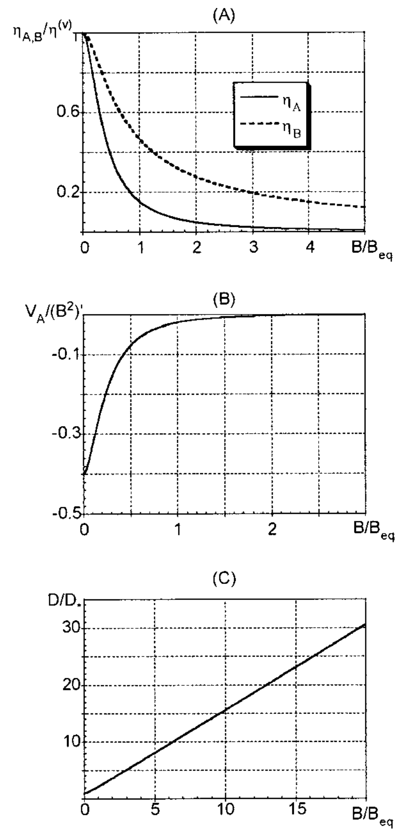

FIG. 2. (a) Nonlinear turbulent magnetic diffusion coefficients; (b) the nonlinear effective velocity; (c) the nonlinear dynamo number for $\eta_T^{(v)}=\eta_T^{(h)}$ and $\varepsilon_\mu^{(v)}=\varepsilon_\mu^{(h)}=0$.

velocity $V_A(B)$ are from the tensor of turbulent diffusion $\eta_{ij}(\mathbf{B})$ and the nonlinear velocity $\mathbf{U}(\mathbf{B})+\mathbf{V}^{(N)}(\mathbf{B})$. The functions $\eta_A(B)$, $\eta_B(B)$, and $V_A(B)$ are given by Eqs. (C11)–(C13) in Appendix C.

The nonlinear dependencies: (a) of the turbulent magnetic diffusion coefficients $\eta_A(B)/\eta_T^{(v)}$ and $\eta_B(B)/\eta_T^{(v)}$; (b) of the effective velocity $V_A(B)/(B^2)'$; and (c) of the nonlinear dynamo number $D(B)/D_*$ are presented in Figs. 1–3. Here $D_*=\alpha_* Gh^3/\eta_*^2$, $D(B)=\alpha^{(v)}(B)Gh^3/[\eta_A(B)\eta_B(B)]$, $\eta_* =\eta_T^{(v)}+(2/3)\varepsilon_\mu^{(v)}$, α_* is the maximum value of the hydrodynamic part of the α effect, h is the disk thickness and $\alpha^{(v)}(B)=\alpha_0^{(v)}\Phi_\alpha(B)$. For simplicity we consider the case $\epsilon=0$. In order to separate the study of the algebraic and dynamic nonlinearities we defined the nonlinear dynamo number $D(B)$ using only the hydrodynamic part of the α effect. We considered three cases: two types of an anisotropic background turbulence ($\varepsilon_\mu^{(v)}=\pm 1.35\eta_T^{(v)}$; $\varepsilon_\mu^{(h)}=0$) without magnetic fluctuations (Fig. 1 and Fig. 3) and an isotropic ($\varepsilon_\mu^{(v)}=\varepsilon_\mu^{(h)}=0$) background turbulence with equipartition of hydrodynamic and magnetic fluctuations (Fig. 2). The negative degree of anisotropy $\varepsilon_\mu^{(v)}$ implies that the vertical (along axis z) size of turbulent elements is less than the horizontal size and positive $\varepsilon_\mu^{(v)}$ means that the horizontal size is less than the vertical size.

Figures 1–3 and the equations for $\eta_A(B)$ and $\eta_B(B)$ show that the toroidal and poloidal magnetic fields have different nonlinear turbulent magnetic diffusion coefficients. In isotropic background turbulence (Fig. 2) the nonlinear effective velocity $V_A(B)$ is negative. The latter implies that it is diamagnetic velocity that results in the field being pushed out from the regions with stronger mean magnetic field. In the anisotropic background turbulence (Fig. 1 and Fig. 3) the nonlinear effective velocity is positive, (i.e., paramagnetic velocity that causes the magnetic field to be concentrated in the regions with stronger field). The sign of $\varepsilon_\mu^{(v)}$ affects the value of $\eta_A(B)$, $\eta_B(B)$ and $V_A(B)$, e.g., for positive parameter of anisotropy the functions $\eta_A(B)$, $\eta_B(B)$, and $V_A(B)$ are larger at least in one order of magnitude than those for negative $\varepsilon_\mu^{(v)}$.

The dependencies of the nonlinear dynamo number $D(B)/D_*$ on the mean magnetic field $B/B_{\rm eq}$ demonstrate that the algebraic nonlinearity alone (i.e., quenching of both, the nonlinear α effect and the nonlinear turbulent diffusion coefficients) cannot saturate the growth of the mean magnetic field (where $B_{\rm eq}=\sqrt{\mu_0\rho}u_0$). Indeed, for anisotropic background turbulence without magnetic fluctuations (Fig. 1 and Fig. 3) the nonlinear dynamo number $D(B)/D_*$ is a nonzero constant for $\beta\gg 1$, i.e., it is independent of β. This is because, for $\beta\gg 1$, the functions $\eta_A\propto 1/\beta$, $\eta_B\propto 1/\beta$, and $\alpha\propto 1/\beta^2$ [see Eqs. (C18)–(C21) in Appendix C]. In the case of isotropic background turbulence with equipartition of hydrodynamic and magnetic fluctuations (Fig. 2) the nonlinear dynamo number $D(B)/D_*\propto\beta$ for $\beta\gg 1$ because in this case the functions $\eta_A\propto 1/\beta^2$, $\eta_B\propto 1/\beta$, and $\alpha\propto 1/\beta^2$ [see Eqs.

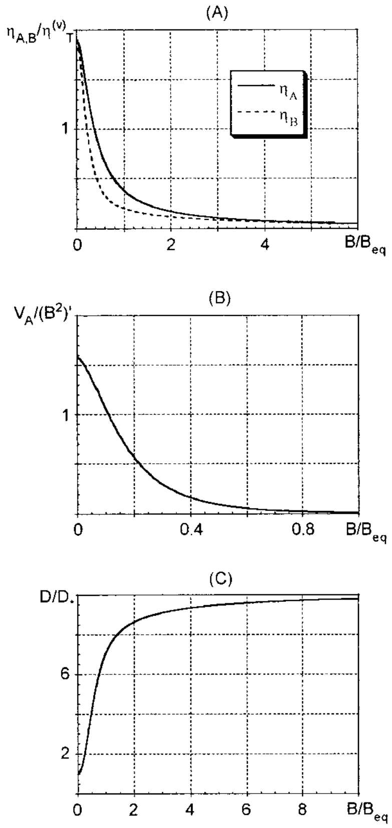

FIG. 3. (a) Nonlinear turbulent magnetic diffusion coefficients; (b) the nonlinear effective velocity; (c) the nonlinear dynamo number for $\eta_T^{(h)}=0$, $\varepsilon_\mu^{(v)}=1.35\eta_T^{(v)}$, and $\varepsilon_\mu^{(h)}=0$.

(C18)–(C21) in Appendix C]. Note that the saturation of the growth of the mean magnetic field can be achieved when the derivative of the nonlinear dynamo number $dD(B)/dB<0$. Thus, the algebraic nonlinearity alone cannot saturate the growth of the mean magnetic field. We will show below that the combined effect of the algebraic and dynamic nonlinearities can limit the growth of the mean magnetic field.

Equation (28) in nondimensional form is given by

$$\frac{\partial \alpha_0^{(h)}}{\partial t}+\frac{\alpha_0^{(h)}}{T}=4\left(\frac{h}{l_0}\right)^2[\eta_B B' A'-(\eta_A A''-V_A A'+\alpha B)B] +[C|\alpha_0^{(v)}(z)|f_\eta(z)\Phi_\alpha(B)\eta_A(B)B^2]', \tag{38}$$

where C is a coefficient, $f_\eta(z)$ describes the inhomogeneity of the turbulent magnetic diffusion, and we define $f(z)=\alpha_0^{(v)}(z)f_\eta(z)$. Here we use the standard dimensionless form of the galactic dynamo equation (see, e.g., Ref. [16]), in particular, the length is measured in units of the disk thickness h, the time is measured in units of $h^2/\eta_T^{(v)}$, and B is measured in units of the equipartition energy $B_{\rm eq}=\sqrt{\mu_0\rho}u_0$. Here u_0 is the characteristic turbulent velocity in the maximum scale l_0 of turbulent motions, $\eta_T^{(v)}=l_0u_0/3$ and $\alpha_0^{(v)}$, $\alpha_0^{(h)}$ and α are measured in units of α_* (the maximum value of the hydrodynamic part of the α effect). For galaxies $h/l_0\sim5$ and $C\sim0.05$–0.1. Nondimensional equations for A and B are given by

$$\partial B/\partial t=(\eta_B B')'-D_0A', \tag{39}$$

$$\partial A/\partial t=\eta_A A''-V_A A'+\alpha B, \tag{40}$$

where $D_0=\alpha_* G h^3/\eta_T^2$ and $B_r=-A'(z)$. In a steady state Eqs. (38)–(40) yield

$$[\eta_B(B)B']^2+2CD_0\Phi_\alpha(B)\eta_A(B)B^2|f(z)|=0, \tag{41}$$

where we used the following boundary conditions $B(z=\pm1)=0$, $B'(z=0)=0$ and $f(z=0)=0$. The solution of Eq. (41) for negative D_0 is given by

$$\int_0^B \chi(\tilde{B})d\tilde{B}=\sqrt{2C|D_0|}\int_{|z|}^1\sqrt{f(\tilde{z})}d\tilde{z}, \tag{42}$$

where $\chi(B)=\eta_B(B)/[\Phi_\alpha(B)\eta_A(B)B^2]^{1/2}$. Consider the case $\epsilon=0$. For $\beta\gg1$ (i.e., for $B\gg1/\sqrt{8}$) the equilibrium mean toroidal magnetic field $B(z)$ is given by

$$B(z)\approx\sqrt{2}C|D_0|\left(\int_{|z|}^1\sqrt{f(\tilde{z})}d\tilde{z}\right)^2, \tag{43}$$

where we used that for $\beta\gg1$ the functions $\eta_A(B)\sim3/5\beta$, $\eta_B(B)\sim2/5\beta$, $\Phi_\alpha(B)\sim2/\beta^2$, and $\chi(B)\sim2/\sqrt{\beta}$. Here for simplicity we considered the case $\varepsilon_\mu^{(v)}=0$. In a steady state $A(z)=-\eta_B(B)B'/|D_0|$. Now we specify the profile of the function $f(z)$, e.g., $f(z)=f_*[\sin(\pi z/2)]^{2k+1}[\cos(\pi z/2)]^2$, where $k=1, 2, 3, \ldots$ and

$$f_*=\left(\frac{2k+3}{2}\right)\left(\frac{2k+3}{2k+1}\right)^{(2k+1)/2}.$$

The function $f(z)$ changes in the interval $0\leqslant f(z)\leqslant1$ and it has a maximum $f(z=z_m)=1$ at $z_m=(2/\pi)\arctan[\sqrt{(2k+1)/2}]$. Equation (43) for this profile $f(z)$ with $k=2$ yields

$$B(z)\approx0.4C|D_0|\{1-[\sin(\pi z/2)]^{7/2}\}^2. \tag{44}$$

Equation (44) describes the equilibrium configuration of the mean toroidal magnetic field. Thus, the saturation of the growth of the mean magnetic field is caused by both, the algebraic and dynamic nonlinearities. The dynamic nonlinearity is determined by the dynamic equation (38), whereas the algebraic nonlinearity implies the nonlinear dependencies of the turbulent magnetic diffusion coefficients $\eta_A(B)$ and $\eta_B(B)$ and of the effective velocity $V_A(B)$ on the mean magnetic field [see Eqs. (C11)–(C13)].

VI. DISCUSSION

In this study we calculated the nonlinear tensor of turbulent magnetic diffusion, the nonlinear κ tensor, the nonlinear effective velocity, and other coefficients defining the mean electromotive force for an anisotropic turbulence. The obtained results were specified for an anisotropic background turbulence with one preferential direction. We found that the turbulent magnetic diffusion coefficients for the toroidal and poloidal magnetic fields are different. We demonstrated that even for a homogeneous turbulence there is the nonlinear effective velocity that can be a diamagnetic or paramagnetic depending on the anisotropy of turbulence and the level of magnetic fluctuations in the background turbulence. The diamagnetic velocity implies that the field is pushed out from the regions with stronger mean magnetic field, while the paramagnetic velocity causes the magnetic field to be concentrated in the regions with stronger field.

Note that dependencies of the α effect, the turbulent magnetic diffusion coefficient and the effective drift velocity on the mean magnetic field for an isotropic turbulence have been found in Refs. [36–38] using a modified second-order correlation approximation. Our results are different from that obtained in Refs. [36–38]. The reason is that in Refs. [36–38] a phenomenological procedure was used. In particular, in the first step of the calculations the nonlinear terms in the magnetohydrodynamic equations were dropped (which is valid for small hydrodynamic and magnetic Reynolds numbers or in a high conductivity limit and small Strouhle and hydrodynamic Reynolds numbers). In the next step of the calculations in Refs. [36–38] it was assumed that $\nu=\eta=l_c^2/\tau_c$, where l_c and τ_c are the correlation length and time of turbulent velocity field, respectively. The latter is valid when the hydrodynamic and magnetic Reynolds numbers are of the order of unity. In the present paper we use a different procedure (the τ approximation) for large hydrodynamic and magnetic Reynolds numbers.

In this study we also demonstrated an important role of two types of nonlinearities (algebraic and dynamic) in the mean-field dynamo. The algebraic nonlinearity is determined by a nonlinear dependence of the mean electromotive force on the mean magnetic field. The dynamic nonlinearity is determined by a differential equation for the magnetic part of the α effect. This equation is a consequence of the conservation of the total magnetic helicity (which includes both, the magnetic helicity of the mean magnetic field and the magnetic helicity of small-scale magnetic fluctuations). We found that at least for the $\alpha\Omega$ axisymmetric dynamo the algebraic nonlinearity alone [i.e., the nonlinear functions $\alpha(B)$, $\eta_A(B)$, $\eta_B(B)$, and $V_A(B)$] cannot saturate the dynamo generated mean magnetic field. The important parameter that characterizes the algebraic nonlinearity is the nonlinear dynamo number $D(B)$. The saturation of the growth of the dynamo generated mean magnetic field by the algebraic nonlinearity alone is possible when the derivative $dD(B)/dB<0$. We found that for the $\alpha\Omega$ axisymmetric dynamo the nonlinear dynamo number $D(B)$ is either a constant or $D(B)\propto B$ for $B>B_{\rm eq}/3$ depending on the model of the background turbulence. Therefore, in this case the algebraic nonlinearity alone cannot saturate the dynamo generated mean magnetic field.

The situation is changed when the dynamic nonlinearity is taken into account. The crucial point is that the dynamic equation for the magnetic part of the α effect (i.e., the dynamic nonlinearity) includes the flux of the magnetic helicity. Without the flux, the total magnetic helicity is conserved locally and the level of the saturated mean magnetic field is very low [8]. The flux of the magnetic helicity results in that the total magnetic helicity is not conserved locally because the magnetic helicity of small-scale magnetic fluctuations is redistributed by a helicity flux. In this case an integral of the total magnetic helicity over the disk is conserved. The equilibrium state is given by a balance between magnetic helicity production and magnetic helicity transport [8]. These two types of the nonlinearities (algebraic and dynamic) result in the equilibrium strength of the mean magnetic field being of the same order as that of the equipartition field $B_{\rm eq}$ (see Sec. V) in agreement with observations of the galactic magnetic fields (see, e.g., [16]).

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APPENDIX A: CALCULATION OF THE MEAN ELECTROMOTIVE FORCE

Let us derive the equations for the second moments. For this purpose we rewrite Eqs. (3) and (4) in a Fourier space and repeat twice the vector multiplication of Eq. (3) by the wave vector $\mathbf{k}$. The result is given by

$$du_i(\mathbf{k},t)/dt=(2P_{ip}(k)-\delta_{ip})\hat{S}_p^{(c)}(b;B)/(\mu_0\rho)+\hat{S}_i^{(b)}(b;B)/(\mu_0\rho)-\tilde{T}_i-\nu k^2u_i-\tilde{F}_i\,, \tag{A1}$$

$$db_i(\mathbf{k},t)/dt=\hat{S}_i^{(b)}(u;B)-\hat{S}_i^{(c)}(u;B)+G_i-\eta k^2b_i\,, \tag{A2}$$

where $\hat{S}_i^{(c)}(a;A)=i\int a_p(\mathbf{k}-\mathbf{Q})Q_pA_i(\mathbf{Q})d\mathbf{Q}$, $\hat{S}_i^{(b)}(a;A)=ik_p\int a_i(\mathbf{k}-\mathbf{Q})A_p(\mathbf{Q})d\mathbf{Q}$, $\tilde{\mathbf{T}}=\mathbf{k}\times(\mathbf{k}\times\mathbf{T})/k^2$, $\tilde{\mathbf{F}}(\mathbf{k},\mathbf{R},t)=\mathbf{k}\times[\mathbf{k}\times\mathbf{F}(\mathbf{k},\mathbf{R})]/k^2\rho$, $P_{ij}(k)=\delta_{ij}-k_{ij}$, δ_{ij} is the Kronecker tensor and $k_{ij}=k_ik_j/k^2$. We use the two-scale approach, i.e., a correlation function

$$\langle u_i(\mathbf{x})u_j(\mathbf{y})\rangle=\int\langle u_i(\mathbf{k}_1)u_j(\mathbf{k}_2)\rangle\exp\{i(\mathbf{k}_1\cdot\mathbf{x}+\mathbf{k}_2\cdot\mathbf{y})\}d\mathbf{k}_1d\mathbf{k}_2=\int f_{ij}(\mathbf{k},\mathbf{K})\exp(i\mathbf{k}\cdot\mathbf{r}+i\mathbf{K}\cdot\mathbf{R})d\mathbf{k}d\mathbf{K}=\int f_{ij}(\mathbf{k},\mathbf{R})\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k},$$

$$f_{ij}(\mathbf{k},\mathbf{R})=\int\langle u_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K},$$

where $\mathbf{R}=(\mathbf{x}+\mathbf{y})/2$, $\mathbf{r}=\mathbf{x}-\mathbf{y}$, $\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2$, $\mathbf{k}=(\mathbf{k}_1-\mathbf{k}_2)/2$, $\mathbf{R}$ and $\mathbf{K}$ correspond to the large scales, and $\mathbf{r}$ and $\mathbf{k}$ to the small ones (see, e.g., Refs. [39,40]). The other second moments have the same form, e.g.,

$$h_{ij}(\mathbf{k},\mathbf{R})=\int \langle b_i(\mathbf{k}+\mathbf{K}/2)b_j(-\mathbf{k}+\mathbf{K}/2)\rangle \times \exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}/\mu_0\rho,$$

$$g_{ij}(\mathbf{k},\mathbf{R})=\int \langle b_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}.$$

The two-scale approach is valid when $(1/B)(dB/dR)\ll l_0^{-1}$, where $B=|\mathbf{B}|$. Now we derive the equations for the correlation functions $f_{ij}(\mathbf{k},\mathbf{R})$, and $h_{ij}(\mathbf{k},\mathbf{R})$, and $g_{ij}(\mathbf{k},\mathbf{R})$

$$\partial f_{ij}/\partial t=i(\mathbf{k}\cdot\mathbf{B})\Phi_{ij}+M_{ij}+F_{ij}-2\nu k^2 f_{ij}, \tag{A3}$$

$$\partial h_{ij}/\partial t=-i(\mathbf{k}\cdot\mathbf{B})\Phi_{ij}+R_{ij}-2\eta k^2 h_{ij}, \tag{A4}$$

$$\partial g_{ij}/\partial t=I_{ij}+C_{ij}-(\nu+\eta)k^2 g_{ij}, \tag{A5}$$

$$I_{ij}=i(\mathbf{k}\cdot\mathbf{B})(f_{ij}-h_{ij})+(1/2)(\mathbf{B}\cdot\boldsymbol{\nabla})(f_{ij}+h_{ij})-f_{pj}B_{i,p}+h_{ip}[2P_{jl}(k)-\delta_{jl}]B_{l,p}-B_{p,q}k_p(f_{ijq}+h_{ijq}), \tag{A6}$$

where $\boldsymbol{\nabla}=\partial/\partial\mathbf{R}$, $f_{ijq}=(1/2)\partial f_{ij}/\partial k_q$, $h_{ijq}=(1/2)\partial h_{ij}/\partial k_q$, and $F_{ij}(\mathbf{k},\mathbf{R})=\langle\tilde{F}_i(\mathbf{k},\mathbf{R})u_j(-\mathbf{k},\mathbf{R})\rangle+\langle u_i(\mathbf{k},\mathbf{R})\tilde{F}_j(-\mathbf{k},\mathbf{R})\rangle$, $B_{i,j}=\partial B_i/\partial R_j$, and

$$\Phi_{ij}(\mathbf{k},\mathbf{R})=[g_{ij}(\mathbf{k},\mathbf{R})-g_{ji}(-\mathbf{k},\mathbf{R})]/\mu_0\rho. \tag{A7}$$

The third moments are given by $M_{ij}(\mathbf{k},\mathbf{R})=-\langle\tilde{T}_i(\mathbf{k})u_j(-\mathbf{k})\rangle-\langle u_i(\mathbf{k})\tilde{T}_j(-\mathbf{k})\rangle$, $R_{ij}(\mathbf{k},\mathbf{R})=\langle\tilde{G}_i(\mathbf{k})b_j(-\mathbf{k})\rangle+\langle b_i(\mathbf{k})\tilde{G}_j(-\mathbf{k})\rangle$ and $C_{ij}(\mathbf{k},\mathbf{R})=\langle\tilde{G}_i(\mathbf{k})u_j(-\mathbf{k})\rangle-\langle b_i(\mathbf{k})\tilde{T}_j(-\mathbf{k})\rangle$.

For the derivation of Eqs. (A3)–(A6) we performed several calculations that are similar to the following, which arose in computing $\partial g_{ij}/\partial t$. The other calculations follow similar lines and are not given here. Let us define $Y_{ij}(\mathbf{k},\mathbf{R})$ by

$$Y_{ij}(\mathbf{k},\mathbf{R})=\int \langle\hat{S}_i^{(b)}(u;B;\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}$$
$$=i\int \langle u_i(\mathbf{k}+\mathbf{K}/2-\mathbf{Q})u_j(-\mathbf{k}+\mathbf{K}/2)\rangle(k_p+K_p/2)B_p(\mathbf{Q})\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}d\mathbf{Q}.$$

Next, we introduce new variables: $\tilde{\mathbf{k}}_1=\mathbf{k}+\mathbf{K}/2-\mathbf{Q}$, $\tilde{\mathbf{k}}_2=-\mathbf{k}+\mathbf{K}/2$ and $\tilde{\mathbf{k}}=(\tilde{\mathbf{k}}_1-\tilde{\mathbf{k}}_2)/2=\mathbf{k}-\mathbf{Q}/2$, $\tilde{\mathbf{K}}=\tilde{\mathbf{k}}_1+\tilde{\mathbf{k}}_2=\mathbf{K}-\mathbf{Q}$. Therefore,

$$Y_{ij}(\mathbf{k},\mathbf{R})=i\int f_{ij}(\mathbf{k}-\mathbf{Q}/2,\mathbf{K}-\mathbf{Q})(k_p+K_p/2)B_p(\mathbf{Q})\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}d\mathbf{Q}. \tag{A8}$$

Since $|\mathbf{Q}|\ll|\mathbf{k}|$ we use the Taylor expansion

$$f_{ij}(\mathbf{k}-\mathbf{Q}/2,\mathbf{K}-\mathbf{Q})\simeq f_{ij}(\mathbf{k},\mathbf{K}-\mathbf{Q})-\frac{1}{2}\frac{\partial f_{ij}(\mathbf{k},\mathbf{K}-\mathbf{Q})}{\partial k_s}Q_s+O(\mathbf{Q}^2), \tag{A9}$$

and the following identities:

$$[f_{ij}(\mathbf{k},\mathbf{R})B_p(\mathbf{R})]_{\mathbf{K}}=\int f_{ij}(\mathbf{k},\mathbf{K}-\mathbf{Q})B_p(\mathbf{Q})d\mathbf{Q}, \tag{A10}$$

$$\nabla_p[f_{ij}(\mathbf{k},\mathbf{R})B_p(\mathbf{R})]=\int iK_p[f_{ij}(\mathbf{k},\mathbf{R})B_p(\mathbf{R})]_{\mathbf{K}}\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}. \tag{A11}$$

Therefore, Eqs. (A8)–(A11) yield

$$Y_{ij}(\mathbf{k},\mathbf{R})\simeq[i(\mathbf{k}\cdot\mathbf{B})+(1/2)(\mathbf{B}\cdot\boldsymbol{\nabla})]f_{ij}(\mathbf{k},\mathbf{R})-k_p f_{ijs}(\mathbf{k},\mathbf{R})B_{p,s}. \tag{A12}$$

In Eqs. (A3) and (A4) we neglected the terms $\propto(\mathbf{B}\cdot\boldsymbol{\nabla})g_{ij}$ and $\propto B_{i,p}g_{pj}$ because they contribute to the modification of the mean Lorentz force by the turbulence effect (see, e.g., Refs. [30,31]). In Eq. (A5) we neglected the second and higher derivatives over $\mathbf{R}$. We also neglected in Eq. (A5) the terms that are of the order of $\mathrm{Rm}^{-1}\boldsymbol{\nabla}(B_i,f_{ij},h_{ij})$ and $\mathrm{Re}^{-1}\boldsymbol{\nabla}(B_i;f_{ij};h_{ij})$. When the mean magnetic field is zero Eq. (A3) reads

$$\partial f_{ij}^{(0)}/\partial t=M_{ij}^{(0)}+F_{ij}^{(0)}-2\nu k^2 f_{ij}^{(0)}. \tag{A13}$$

We assume that F_{ij} is not changed during the generation of the mean magnetic field, i.e., $F_{ij}=F_{ij}^{(0)}$. This implies an assumption of a constant power of the source of turbulence.

Now we split all correlation functions (i.e., $f_{ij},h_{ij},g_{ij},\Phi_{ij}$) into two parts, e.g., $f_{ij}=f_{ij}^{(N)}+f_{ij}^{(S)}$, where $f_{ij}^{(N)}=[f_{ij}(\mathbf{k},\mathbf{R})+f_{ij}(-\mathbf{k},\mathbf{R})]/2$ and $f_{ij}^{(S)}=[f_{ij}(\mathbf{k},\mathbf{R})-f_{ij}(-\mathbf{k},\mathbf{R})]/2$. Next, we use τ approximation that is determined by Eqs. (8)–(10). We assume that $\eta k^2\ll\tau^{-1}$ and $\nu k^2\ll\tau^{-1}$ for the inertial range of turbulent fluid flow. We also assume that the characteristic time of variation of the mean magnetic field $\mathbf{B}$ is substantially longer than the correlation time $\tau(k)$ for all turbulence scales. Thus, Eqs. (A3)–(A5) yield

$$f_{ij}^{(N)}\approx f_{ij}^{(0N)}+i\tau(\mathbf{k}\cdot\mathbf{B})\Phi_{ij}^{(S)}, \tag{A14}$$

$$h_{ij}^{(N)}\approx h_{ij}^{(0N)}-i\tau(\mathbf{k}\cdot\mathbf{B})\Phi_{ij}^{(S)}, \tag{A15}$$

$$f_{ij}^{(S)}\approx f_{ij}^{(0S)}+i\tau(\mathbf{k}\cdot\mathbf{B})\Phi_{ij}^{(N)}, \tag{A16}$$

$$g_{ij}\approx\tau I_{ij}, \tag{A17}$$

where $\psi=2(\mathbf{k}\cdot\mathbf{B}\tau)^2/\mu_0\rho$, $k_{ij}=k_ik_j/k^2$, $f_{ij}^{(0N)}$ and $f_{ij}^{(0S)}$ describe the nonhelical and helical tensors of the background turbulence. The tensor $h_{ij}^{(S)}$ is determined by an evolutionary equation (see, e.g., Refs. [15,23,32,24,2,25,8] and Sec. III C). Now we calculate $\Phi_{ij}^{(N)}$ and $\Phi_{ij}^{(S)}$. The definition of Φ_{ij}, given by Eqs. (A7) and (A17) yields

$$\Phi_{ij}(\mathbf{k},\mathbf{R})\approx\tau(\mu_0\rho)^{-1}[I_{ij}(\mathbf{k},\mathbf{R})-I_{ji}(-\mathbf{k},\mathbf{R})]. \quad (A18)$$

Substituting Eq. (A6) into Eq. (A18) and using Eqs. (5) and (6) we obtain

$$\Phi_{ij}\approx\tau(\mu_0\rho)^{-1}\{2i(\mathbf{k}\cdot\mathbf{B})(f_{ij}-h_{ij})-B_{i,p}(f_{pj}+h_{pj}) + B_{j,p}(f_{ip}+h_{ip})+2B_{l,p}(h_{pj}k_{li}-h_{ip}k_{lj})\}. \quad (A19)$$

Now using Eqs. (A14)–(A16) and Eq. (A19) we get

$$\Phi_{ij}^{(N)}\approx\tau(1+\psi)^{-1}(\mu_0\rho)^{-1}\{2i(\mathbf{k}\cdot\mathbf{B})(f_{ij}^{(0S)}-h_{ij}^{(S)}) + B_{j,p}(f_{pi}^{(0N)}+h_{pi}^{(0N)})-B_{i,p}(f_{pj}^{(0N)}+h_{pj}^{(0N)}) + 2B_{p,l}(h_{jl}^{(N)}k_{pi}-h_{il}^{(N)}k_{pj})\}, \quad (A20)$$

$$\Phi_{ij}^{(S)}\approx 2i\tau(1+2\psi)^{-1}(\mu_0\rho)^{-1}(\mathbf{k}\cdot\mathbf{B})(f_{ij}^{(0N)}-h_{ij}^{(0N)}) + O(B_{i,j}). \quad (A21)$$

To calculate the electromotive force we do not take into account the second- and higher-orders spatial derivatives of the mean magnetic field. This implies that in the tensor b_{ijk} (and, therefore, in the tensor $\Phi_{ij}^{(S)}$) we neglect the first- and higher-orders spatial derivatives of the mean magnetic field [see Eqs. (11), (A21), and (A23) below].

Note that Eqs. (A14) and (A15) yield

$$f_{ij}^{(N)}+h_{ij}^{(N)}\approx f_{ij}^{(0N)}+h_{ij}^{(0N)}. \quad (A22)$$

This is in agreement with the fact that a uniform mean magnetic field performs no work on the turbulence. It can only redistribute the energy between hydrodynamic fluctuations and magnetic fluctuation. Analysis in Ref. [31] showed that a change of the total energy (kinetic and magnetic) of fluctuations caused by a nonuniform mean magnetic field is of the order $\sim\tau\eta_T^{(v)}\Delta B^2$. In the case of a very strong mean magnetic field Eq. (A22) can be violated.

Using Eqs. (A14)–(A16) and (A20)–(A21) we calculate the electromotive force $\mathcal{E}_i(\mathbf{r}=0)=\int\mathcal{E}_i(\mathbf{k})d\mathbf{k}$, where the Fourier component $\mathcal{E}_i(\mathbf{k})=(\mu_0\rho/2)\varepsilon_{imn}\Phi_{nm}^{(N)}(\mathbf{k})$, and ε_{ijk} is the Levi-Civita tensor. The electromotive force is given by Eqs. (11)–(13). Substituting Eq. (A15) into Eq. (13) we get

$$b_{ijk}=\int \tau(1+\psi)^{-1}\{\varepsilon_{ijn}(f_{kn}^{(0N)}+h_{kn}^{(0N)})-2\varepsilon_{imn}k_{mj}[h_{nk}^{(0N)} - i\tau(\mathbf{k}\cdot\mathbf{B})\Phi_{nk}^{(S)}]\}d\mathbf{k}. \quad (A23)$$

The integration in $\mathbf{k}$ space in Eq. (A23) yields

$$b_{ijk}=\varepsilon_{ijn}\lambda_{nk}^{(P)}(\beta)+2\varepsilon_{inm}\zeta_{nkjm}^{(C)}(\beta). \quad (A24)$$

Hereafter we use the following definitions:

$$X_{ijk\ldots}^{(C)}(\beta)=X_{ijk\ldots}^{(v)}(\beta)-X_{ijk\ldots}^{(v)}(\sqrt{2}\beta)+X_{ijk\ldots}^{(h)}(\sqrt{2}\beta), \quad (A25)$$

$$X_{ijk\ldots}^{(M)}(\beta)=X_{ijk\ldots}^{(v)}(\beta)-X_{ijk\ldots}^{(h)}(\beta), \quad (A26)$$

$$X_{ijk\ldots}^{(P)}(\beta)=X_{ijk\ldots}^{(v)}(\beta)+X_{ijk\ldots}^{(h)}(\beta), \quad (A27)$$

and

$$\lambda_{ij}^{(a)}(\beta)=\int\frac{c_{ij}(\mathbf{k})\tau(k)}{1+\psi(\beta,\mathbf{k})}d\mathbf{k}, \quad (A28)$$

$$\zeta_{ijmn}^{(a)}(\beta)=\int\frac{c_{ij}(\mathbf{k})\tau(k)}{1+\psi(\beta,\mathbf{k})}k_{mn}d\mathbf{k}, \quad (A29)$$

and $\beta_i=4B_i/(u_0\sqrt{2\mu_0\rho})$, $\psi(\beta,\mathbf{k})=[(\boldsymbol{\beta}\cdot\mathbf{k})u_0\tau/2]^2$, $c_{ij}=f_{ij}^{(0N)}$ for $\lambda_{ij}^{(v)}$, and $c_{ij}=h_{ij}^{(0N)}$ for $\lambda_{ij}^{(h)}$.

For the calculation of the tensor b_{ijk} we specified a model of the background turbulence (i.e., turbulence with zero mean magnetic field). The turbulent velocity and magnetic fields of the background turbulence are determined by Eq. (17). To integrate over the angles in $\mathbf{k}$ space in Eqs. (A28) and (A29) we use the following identities:

$$\int\frac{k_{ij}\sin\theta}{1+a\cos^2\theta}d\theta d\varphi=\bar{A}_1\delta_{ij}+\bar{A}_2\beta_{ij}, \quad (A30)$$

$$\int\frac{k_{ijmn}\sin\theta}{1+a\cos^2\theta}d\theta d\varphi=\bar{C}_1(\delta_{ij}\delta_{mn}+\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) + \bar{C}_2\beta_{ijmn}+\bar{C}_3(\delta_{ij}\beta_{mn}+\delta_{im}\beta_{jn} + \delta_{in}\beta_{jm}+\delta_{jm}\beta_{in}+\delta_{jn}\beta_{im} + \delta_{mn}\beta_{ij}), \quad (A31)$$

where $a=[\beta u_0 k\tau(k)/2]^2$, $\hat{\beta}_i=\beta_i/\beta$, $\beta_{ij}=\hat{\beta}_i\hat{\beta}_j$, $\beta_{ijk\ldots}=\hat{\beta}_i\hat{\beta}_j\hat{\beta}_k\ldots$, and

$$\bar{A}_1=\frac{2\pi}{a}\left[(a+1)\frac{\arctan(\sqrt{a})}{\sqrt{a}}-1\right],$$

$$\bar{A}_2=-\frac{2\pi}{a}\left[(a+3)\frac{\arctan(\sqrt{a})}{\sqrt{a}}-3\right],$$

$$\bar{C}_1=\frac{\pi}{2a^2}\left[(a+1)^2\frac{\arctan(\sqrt{a})}{\sqrt{a}}-\frac{5a}{3}-1\right],$$

$$\bar{C}_2=\frac{\pi}{2a^2}\left[(3a^2+30a+35)\frac{\arctan(\sqrt{a})}{\sqrt{a}}-\frac{55a}{3}-35\right],$$

$$\bar{C}_3=-\frac{\pi}{2a^2}\left[(a^2+6a+5)\frac{\arctan(\sqrt{a})}{\sqrt{a}}-\frac{13a}{3}-5\right].$$

To integrate over k in Eqs. (A28) and (A29) we use the Kolmogorov spectrum of the background turbulence, i.e., $\tau f_{pp}^{(0N)}(\mathbf{k}) = \eta_T^{(v)} \varphi(k)$, $\tau h_{pp}^{(0N)}(\mathbf{k}) = \eta_T^{(h)} \varphi(k)$, and $\mu_{ij}^{(a)}(\mathbf{k}) = \mu_{ij}^{(a)}(\mathbf{R}) \varphi(k)/3$, where $\varphi(k) = (\pi k^2 k_0)^{-1}(k/k_0)^{-7/3}$, $\tau(k) = 2\tau_0(k/k_0)^{-2/3}$, where $k_0 \leqslant k \leqslant k_d$, $k_0 = l_0^{-1}$, l_0 is the maximum scale of turbulent motions and $k_d = k_0 \mathrm{Re}^{3/4}$ is determined by the Kolmogorov's viscous scale of turbulence. The integration in $\mathbf{k}$ space in Eqs. (A28) and (A29) yields

$$\lambda_{ij}^{(a)}(\beta) = \Lambda_{ij}^{(a)}(\beta) + \hat{\beta}_i[\gamma^{(a)}(\beta)\hat{\beta}_j + \Psi_2(\beta)\mu_{jn}^{(a)}\hat{\beta}_n], \tag{A32}$$

$$\zeta_{ijmn}^{(a)}(\beta) = \xi_{ijmn}^{(a)}(\beta) + \hat{\beta}_n[U_{ijm}^{(a)}(\beta) + \Gamma^{(a)}(\beta)\delta_{ij}\hat{\beta}_m], \tag{A33}$$

where

$$\Lambda_{ij}^{(a)}(\beta) = \Psi_1(\beta)\mu_{ij}^{(a)} + \Psi_2(\beta)\mu_{in}^{(a)}\beta_{nj} + \delta_{ij}\{[A_1(\beta) + (1/2)A_2(\beta)]\eta_T^{(a)} + (1/4)\Psi_3(\beta)\mu_\beta^{(a)}\}, \tag{A34}$$

$$\gamma^{(a)}(\beta) = (5/12)C_2(\beta)\mu_\beta^{(a)} - (1/2)A_2(\beta)\eta_T^{(a)}, \tag{A35}$$

$$\Gamma^{(a)}(\beta) = (5/12)C_2(\beta)\mu_\beta^{(a)} + (1/2)A_2(\beta)\eta_T^{(a)}, \tag{A36}$$

$$U_{ijm}^{(a)}(\beta) = (5/6)\{[A_2(\beta) - C_3(\beta)]\mu_{ij}^{(a)}\hat{\beta}_m - C_3(\beta)(\mu_{im}^{(a)}\hat{\beta}_j + \mu_{ip}^{(a)}\hat{\beta}_p\delta_{mj} - \mu_{mp}^{(a)}\hat{\beta}_p\delta_{ij}) - C_2(\beta)\mu_{ip}^{(a)}\beta_{pjm}\}, \tag{A37}$$

$$\begin{aligned}\xi_{ijmn}^{(a)}(\beta) = {}&(1/2)\delta_{mn}\{[A_1(\beta)\eta_T^{(a)} + (5/6)C_3(\beta)\mu_\beta^{(a)}]\delta_{ij} \\ &+ (5/3)[A_1(\beta) - C_1(\beta)]\mu_{ij}^{(a)} \\ &- (5/3)C_3(\beta)\mu_{ip}^{(a)}\beta_{pj}\} + (5/6)\{\delta_{ij}[C_1(\beta)\mu_{mn}^{(a)} \\ &+ C_3(\beta)\mu_{np}^{(a)}\beta_{pm}] - C_1(\beta)\mu_{im}^{(a)}\delta_{jn} \\ &- C_3(\beta)\mu_{ip}^{(a)}\beta_{pm}\delta_{jn}\},\end{aligned} \tag{A38}$$

and $\mu_\beta^{(a)} = \mu_{ps}^{(a)}\beta_{sp}$, $\Psi_1(\beta) = (5/6)[A_1(\beta) + A_2(\beta) + C_1(\beta)]$, $\Psi_2(\beta) = (5/6)[C_3(\beta) - A_2(\beta)]$, $\Psi_3(\beta) = (5/3)[A_2(\beta) + C_3(\beta)]$. The functions $A_n(\beta) = \int_{k_0}^\infty \bar{A}_n(a)\varphi(k)k^2 dk = (3\beta^4/\pi)\int_\beta^\infty [\bar{A}_n(X^2)/X^5]dX$ and similarly for $C_n(\beta)$, where $a = [\beta u_0 k \tau(k)/2]^2 = X^2 = \beta^2(k/k_0)^{2/3}$, and we took into account that the inertial range of the turbulence exists in the scales: $l_d \leqslant r \leqslant l_0$. Here the maximum scale of the turbulence $l_0 \ll L_B$, and $l_d = l_0/\mathrm{Re}^{3/4}$ is the viscous scale of turbulence, and L_B is the characteristic scale of variations of the nonuniform mean magnetic field. For very large Reynolds numbers $k_d = l_d^{-1}$ is very large and the turbulent hydrodynamic and magnetic energies are very small in the viscous dissipative range of the turbulence $0 \leqslant r \leqslant l_d$. Thus we integrated in A_n over k from $k_0 = l_0^{-1}$ to ∞. The functions $A_n(\beta)$ and $C_n(\beta)$ are given in Appendix B. In Eqs. (A33)–(A38) we omitted terms that are symmetric in indexes i and n because after multiplication $\zeta_{ijmn}^{(a)}(\beta)$ by ε_{lin} these symmetric terms vanish [see Eq. (A24)].

In order to extract terms $\propto \varepsilon_{ijm}\hat{\beta}_m$, which contribute to the nonlinear diamagnetic and paramagnetic velocities, we split b_{ijk} into two parts, i.e., $b_{ijk} = b_{ijk}^{(1)} + b_{ijk}^{(2)}$, where

$$b_{ijk}^{(1)} = \varepsilon_{inm}\hat{\beta}_m\{\delta_{jn}[\gamma^{(P)}(\beta)\hat{\beta}_k + \Psi_2(\beta)\mu_{kp}^{(P)}\hat{\beta}_p] + 2[\Gamma^{(C)}(\beta)\delta_{nk}\hat{\beta}_j + U_{nkj}^{(C)}(\beta)]\}, \tag{A39}$$

$$b_{ijk}^{(2)} = \varepsilon_{ijn}\Lambda_{nk}^{(P)}(\beta) + 2\varepsilon_{inm}\xi_{nkjm}^{(C)}(\beta) \tag{A40}$$

[see the definitions given by Eqs. (A25)–(A27)]. Next, we calculate $b_{ijk}B_{j,k}$. Using Eqs. (11), (A39), and (A40) we also split the electromotive force into two parts

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(1)} + \boldsymbol{\mathcal{E}}^{(2)}, \tag{A41}$$

$$\mathcal{E}_i^{(1)} = b_{ijk}^{(1)} B_{j,k}, \tag{A42}$$

$$\mathcal{E}_i^{(2)} = a_{ij}B_j + b_{ijk}^{(2)} B_{j,k}. \tag{A43}$$

Using Eqs. (A39) and (A42) we obtain

$$\mathcal{E}_i^{(1)} = (\mathbf{V}^{(N)} \times \mathbf{B})_i - \eta_{ij}^{(1)}(\boldsymbol{\nabla} \times \mathbf{B})_j, \tag{A44}$$

where

$$V_i^{(N)}(\mathbf{B}) = \frac{1}{2B^2}[\gamma^{(P)}(\beta) + 2\Gamma^{(C)}(\beta)]\nabla_i B^2 + \frac{1}{B}[2U_{ikj}^{(C)}(\beta) + \Psi_2(\beta)\mu_{kp}^{(P)}\hat{\beta}_p\delta_{ij}]\nabla_k B_j, \tag{A45}$$

$$\eta_{ij}^{(1)} = \gamma^{(P)}(\beta)P_{ij}(\beta), \tag{A46}$$

and $P_{ij}(\beta) = \delta_{ij} - \beta_{ij}$. For the calculation of the terms $\propto \gamma^{(P)}(\beta)$ in these equations we used an identity $\varepsilon_{imn}\beta_{np}B_{m,p} \equiv -[\mathbf{B} \times (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}]_i/B^2 = -[\mathbf{B} \times \boldsymbol{\nabla}(B^2/2)]_i/B^2 - P_{ip}(\beta)(\boldsymbol{\nabla} \times \mathbf{B})_p$, which follows from the formula $(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B} = (1/2)\boldsymbol{\nabla}B^2 - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})$. Following Ref. [21] we use an identity $B_{j,k} = (\partial\hat{B})_{jk} - \varepsilon_{jkl}(\boldsymbol{\nabla} \times \mathbf{B})_l/2$ in order to rewrite Eq. (A43) in the form

$$\mathcal{E}_i^{(2)} = \alpha_{ij}B_j + (\mathbf{U} \times \mathbf{B})_i - \eta_{ij}^{(2)}(\boldsymbol{\nabla} \times \mathbf{B})_j - \kappa_{ijk}(\partial\hat{B})_{jk}, \tag{A47}$$

where

$$\eta_{ij}^{(2)} = (\varepsilon_{ikp}b_{jkp}^{(2)} + \varepsilon_{jkp}b_{ikp}^{(2)})/4, \tag{A48}$$

$$\kappa_{ijk}(\mathbf{B}) = -(b_{ijk}^{(2)} + b_{ikj}^{(2)})/2. \tag{A49}$$

Using Eqs. (A44) and (A47) we obtain the equation for the electromotive force $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(1)} + \boldsymbol{\mathcal{E}}^{(2)}$ which is given by Eq. (26). The tensor of turbulent magnetic diffusion

$$\eta_{ij}(\mathbf{B}) = \eta_{ij}^{(1)} + \eta_{ij}^{(2)} \tag{A50}$$

is given by

$$\eta_{ij}(\mathbf{B})=\delta_{ij}\{A_1(\beta)\,\eta_T^{(P)}+(5/12)[C_2(\beta)+2C_3(\beta)]\mu_\beta^{(P)} -[A_1\eta_T]^{(C)}-(5/6)[C_3\mu_\beta]^{(C)}\}-(1/4) \times[2\Psi_1(\beta)\mu_{ij}^{(P)}+\Psi_2(\beta)\overline{\mu}_{ij}^{(P)}]+(5/6)[(A_1 +C_1)\mu_{ij}]^{(C)}+(5/12)[C_3\overline{\mu}_{ij}]^{(C)} +(1/2)\beta_{ij}[A_2(\beta)\,\eta_T^{(P)}-(5/6)C_2(\beta)\mu_\beta^{(P)}], \tag{A51}$$

where $\overline{\mu}_{ij}^{(a)}=\mu_{in}^{(a)}\beta_{nj}+\beta_{in}\mu_{nj}^{(a)}$, and we used Eqs. (A46), (A48), and the definitions (A25)–(A27). In particular, $[X]^{(C)}(\beta)=X^{(v)}(\beta)-X^{(v)}(\sqrt{2}\beta)+X^{(h)}(\sqrt{2}\beta)$ that implies, e.g., $[A_1\eta_T]^{(C)}=A_1(\beta)\,\eta_T^{(v)}-A_1(\sqrt{2}\beta)\,\eta_T^{(v)}+A_1(\sqrt{2}\beta)\,\eta_T^{(h)}$.

Using Eqs. (A40) and (A49) we calculate $\kappa_{ijk}(\mathbf{B})$

$$\kappa_{ijk}(\mathbf{B})=-(1/2)[\Psi_1(\beta)\hat{L}_{ijk}^{(P)}+\Psi_2(\beta)\hat{N}_{ijk}^{(P)}]+(5/6)[(A_1 -3C_1)\hat{L}_{ijk}]^{(C)}-(5/2)[C_3\hat{N}_{ijk}]^{(C)}, \tag{A52}$$

where $\hat{L}_{ijk}^{(a)}=\varepsilon_{ijn}\mu_{nk}^{(a)}+\varepsilon_{ikn}\mu_{nj}^{(a)}$, $\hat{N}_{ijk}^{(a)}=\mu_{np}^{(a)}(\varepsilon_{ijn}\beta_{pk}+\varepsilon_{ikn}\beta_{pj})$, and $[C_3\hat{N}_{ijk}]^{(C)}=C_3(\beta)\hat{N}_{ijk}^{(v)}-C_3(\sqrt{2}\beta)(\hat{N}_{ijk}^{(v)}-\hat{N}_{ijk}^{(h)})$ and similarly for $[(A_1-3C_1)\hat{L}_{ijk}]^{(C)}$ [see Eq. (A25)].

The asymptotic formulas for the nonlinear coefficients defining the mean electromotive force for $\beta\ll 1$ are given by

$$\eta_{ij}(\mathbf{B})=\delta_{ij}\eta_T^{(v)}-\frac{1}{2}\mu_{ij}^{(M)}-\frac{2}{5}\beta^2\bigg[\delta_{ij}\bigg\{2\,\eta_T^{(v)}-\eta_T^{(h)} +\frac{5}{21}(2\mu_\beta^{(v)}-\mu_\beta^{(h)})\bigg\}+\frac{5}{42}(5\mu_{ij}^{(h)}-19\mu_{ij}^{(v)}+2\overline{\mu}_{ij}^{(v)} +5\overline{\mu}_{ij}^{(h)})+\beta_{ij}\eta_T^{(P)}\bigg], \tag{A53}$$

$$\alpha_{ij}^{(v)}(\mathbf{B})=\alpha_0^{(v)}\delta_{ij}+\nu_{ij}-(2/5)\beta^2\{\delta_{ij}[3\alpha_0^{(v)}+\nu_\beta\{1+(8/7)\epsilon\}] +\nu_{ij}[2-(9/7)\epsilon]\}, \tag{A54}$$

$$\alpha_{ij}^{(h)}(\mathbf{B})=\alpha_0^{(h)}(\mathbf{B})(1-3\beta^2/5)\,\delta_{ij}, \tag{A55}$$

and for $\beta\gg 1$ they are given by

$$\eta_{ij}(\mathbf{B})=\frac{3\pi}{5\beta}\bigg[\delta_{ij}\bigg\{\eta_T^{(M)}/\sqrt{2}+\eta_T^{(h)}+\frac{5}{48}[\mu_\beta^{(v)}(3-\sqrt{2}) +\mu_\beta^{(h)}(\sqrt{2}+1)]\bigg\}+\frac{5}{48}\bigg\{\mu_{ij}^{(v)}(9-5\sqrt{2})+\mu_{ij}^{(h)}(5\sqrt{2} -1)-\frac{3}{10}[\overline{\mu}_{ij}^{(v)}(5-\sqrt{2})+\overline{\mu}_{ij}^{(h)}(3+\sqrt{2})]\bigg\} -\frac{1}{2}\beta_{ij}\bigg(\eta_T^{(P)}+\frac{5}{8}\mu_\beta^{(P)}\bigg)\bigg], \tag{A56}$$

$$\alpha_{ij}^{(v)}(\mathbf{B})=-(3\pi/10\beta)[\delta_{ij}(1-\epsilon)\nu_\beta-\nu_{ij}(1+9\epsilon)] +(2\alpha_0^{(v)}/\beta^2)\,\delta_{ij}, \tag{A57}$$

$$\alpha_{ij}^{(h)}(\mathbf{B})=(3\pi/2\beta^2)\,\alpha_0^{(h)}(\mathbf{B})\,\delta_{ij}. \tag{A58}$$

The asymptotic formulas for the tensor κ_{ijk} for $\beta\ll 1$ and $\beta\gg 1$ are given by Eqs. (21) and (22).

APPENDIX B: THE FUNCTIONS $A_\alpha(\beta)$ AND $C_\alpha(\beta)$

The functions $A_\alpha(\beta)$ and $C_\alpha(\beta)$ are given by

$$A_1(\beta)=\frac{6}{5}\left[\frac{\arctan\beta}{\beta}\left(1+\frac{5}{7\beta^2}\right)+\frac{1}{14}L(\beta)-\frac{5}{7\beta^2}\right],$$

$$A_2(\beta)=-\frac{6}{5}\left[\frac{\arctan\beta}{\beta}\left(1+\frac{15}{7\beta^2}\right)-\frac{2}{7}L(\beta)-\frac{15}{7\beta^2}\right],$$

$$C_1(\beta)=\frac{3}{10}\left[\frac{\arctan\beta}{\beta}\left(1+\frac{10}{7\beta^2}+\frac{5}{9\beta^4}\right)+\frac{2}{63}L(\beta) -\frac{235}{189\beta^2}-\frac{5}{9\beta^4}\right],$$

$$C_2(\beta)=\frac{3}{2}\left[\frac{\arctan\beta}{\beta}\left(\frac{3}{5}+\frac{30}{7\beta^2}+\frac{35}{9\beta^4}\right)+\frac{16}{315}L(\beta) -\frac{565}{189\beta^2}-\frac{35}{9\beta^4}\right],$$

$$C_3(\beta)=-\frac{3}{2}\left[\frac{\arctan\beta}{\beta}\left(\frac{1}{5}+\frac{6}{7\beta^2}+\frac{5}{9\beta^4}\right)-\frac{8}{315}L(\beta) -\frac{127}{189\beta^2}-\frac{5}{9\beta^4}\right],$$

where $L(\beta)=1-2\beta^2+2\beta^4\ln(1+\beta^{-2})$. For $\beta\ll 1$ these functions are given by

$$A_1(\beta)\sim 1-(2/5)\beta^2,\quad A_2(\beta)\sim-(4/5)\beta^2,$$

$$C_1(\beta)\sim(1/5)[1-(2/7)\beta^2],\quad C_2(\beta)\sim -(32/105)\beta^4\ln\beta,\quad C_3(\beta)\sim-(4/35)\beta^2,$$

and for $\beta\gg 1$ they are given by

$$A_1(\beta)\sim 3\pi/5\beta-2/\beta^2,\quad A_2(\beta)\sim-3\pi/5\beta+4/\beta^2,$$

$$C_1(\beta)\sim 3\pi/20\beta,\quad C_2(\beta)\sim 9\pi/20\beta,\quad C_3(\beta)\sim -3\pi/20\beta.$$

Since the function $A_1(\beta)+A_2(\beta)\sim O(\beta^{-2})$ for $\beta\gg 1$ (it describes an isotropic part of the α effect) we took into account in the functions $A_1(\beta)$ and $A_2(\beta)$ the terms that are of the order of $\sim O(\beta^{-2})$. Here we also used that for $\beta\ll 1$ the function $L(\beta)\sim 1-2\beta^2-4\beta^4\ln\beta$, and for $\beta\gg 1$ the function $L(\beta)\sim 2/3\beta^2$.

APPENDIX C: DERIVATION OF THE NONLINEAR DEPENDENCIES $\eta_A(B)$, $\eta_B(B)$ AND $V_A(B)$

Now we consider an anisotropic background turbulence with one preferential direction, say along unit vector **e**, where $\mathbf{e}\cdot\hat{\boldsymbol{\beta}}=0$. In this case

$$\mathbf{V}^{(N)}=[V^{(1)}\boldsymbol{\nabla}B^2+V^{(2)}\mathbf{e}(\mathbf{e}\cdot\boldsymbol{\nabla})B^2+V^{(3)}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}]/B^2, \tag{C1}$$

$$\mathbf{U}=[U^{(1)}\boldsymbol{\nabla}B^2+U^{(2)}\mathbf{e}(\mathbf{e}\cdot\boldsymbol{\nabla})B^2+U^{(3)}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}]/B^2, \tag{C2}$$

$$\kappa_{ijk}(\partial\hat{B})_{jk}=-\{\tilde{W}[\boldsymbol{\nabla}B^2+2(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}]\times\mathbf{B}-M_\kappa\mathbf{e}\times(\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{B}\}_i/B^2, \tag{C3}$$

where

$$V^{(1)}=-(1/4)\{A_2(\beta)\,\eta_T^{(P)}+(5/18)C_2(\beta)\varepsilon_\mu^{(P)}+(5/9)[(C_2+2A_2)\varepsilon_\mu]^{(C)}-(1/2)[A_2\,\eta_T]^{(C)}\},$$

$$V^{(2)}=(5/6)[(A_2-C_3)\varepsilon_\mu]^{(C)},\quad V^{(3)}=(5/9)[C_3\varepsilon_\mu]^{(C)}-(1/3)\Psi_2(\beta)\varepsilon_\mu^{(P)},$$

$$U^{(1)}=-(\sqrt{2}\beta/48)\Psi(\sqrt{2}\beta),\quad U^{(2)}=-(\sqrt{2}\beta/4)\Psi_1'(\sqrt{2}\beta)\varepsilon_\mu^{(M)},$$

$$U^{(3)}=(1/6)\Psi_2(\sqrt{2}\beta)\varepsilon_\mu^{(M)},\quad \tilde{W}=-(1/12)\{\Psi_2(\beta)\varepsilon_\mu^{(P)}+5[C_3\varepsilon_\mu]^{(C)}\}.$$

In order to derive Eq. (C3) we used the following identities:

$$L_{ijk}^{(a)}(\partial\hat{B})_{jk}=-\varepsilon_\mu^{(a)}[\mathbf{e}\times(\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{B}]_i,$$

$$N_{ijk}^{(a)}(\partial\hat{B})_{jk}=-(1/6B^2)\varepsilon_\mu^{(a)}[(\boldsymbol{\nabla}B^2+2(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B})\times\mathbf{B}]_i.$$

Using Eqs. (C1)–(C3) and (A51) we calculate the functions M_η, M_e, M_β, M_κ, $M_V^{(1)}$, and $M_V^{(2)}$ in Eqs. (30)–(32),

$$M_\eta=A_1(\beta)\,\eta_T^{(P)}+(5/36)[A_1(\beta)+4A_2(\beta)+C_1(\beta)-C_2(\beta)-5C_3(\beta)]\varepsilon_\mu^{(P)}-(5/18)[(C_1+A_1)\varepsilon_\mu]^{(C)}-[A_1\,\eta_T]^{(C)}+(1/6)\Psi_2(\sqrt{2}\beta)\varepsilon_\mu^{(M)}, \tag{C4}$$

$$M_\kappa=(1/2)\Psi_1(\beta)\varepsilon_\mu^{(P)}+(5/6)[(3C_1-A_1)\varepsilon_\mu]^{(C)}, \tag{C5}$$

$$M_\beta=(1/6)\{3A_2(\beta)\,\eta_T^{(P)}+[(5/6)C_2(\beta)+4\Psi_2(\beta)]\varepsilon_\mu^{(P)}-\Psi_2(\sqrt{2}\beta)\varepsilon_\mu^{(M)}\}, \tag{C6}$$

$$M_e=-(1/2)\Psi_1(\beta)\varepsilon_\mu^{(P)}+(5/6)[(C_1+A_1)\varepsilon_\mu]^{(C)}, \tag{C7}$$

$$M_V^{(1)}\equiv V^{(1)}+U^{(1)}+2W+(1/2)(V^{(3)}+U^{(3)})=-(1/4)A_2(\beta)\,\eta_T^{(P)}-(5/72)[C_2(\beta)+4C_3(\beta)-4A_2(\beta)]\varepsilon_\mu^{(P)}+(1/12)\Psi_2(\sqrt{2}\beta)\varepsilon_\mu^{(M)}+(5/36)\times\{[A_2\,\eta_T]^{(C)}-[(C_2+2A_2+4C_3)\varepsilon_\mu]^{(C)}\}-(\sqrt{2}\beta/48)\Psi(\sqrt{2}\beta), \tag{C8}$$

$$M_V^{(2)}\equiv V^{(2)}+U^{(2)}=(5/6)[(A_2-C_3)\varepsilon_\mu]^{(C)}-(\sqrt{2}\beta/4)\Psi_1'(\sqrt{2}\beta)\varepsilon_\mu^{(M)}. \tag{C9}$$

Now we take into account that $\mathbf{V}^{(N)}$, **U**, and $\boldsymbol{\kappa}$ contribute into the tensor η_{ij}. This implies that in order to calculate M_η, M_e, and M_β we perform the change

$$\eta_{ij}\to\eta_{ij}+P_{ij}(\beta)[V^{(3)}+U^{(3)}+2W], \tag{C10}$$

where the second term in Eq. (C10) [which is proportional to $P_{ij}(\beta)$] describes a contribution $\mathbf{V}^{(N)}$, **U**, and $\boldsymbol{\kappa}$ into the tensor η_{ij}. Using Eqs. (30)–(32) and (C4)–(C9) we calculate the functions $\eta_A(B)$, $\eta_B(B)$, and $V_A(B)$

$$\eta_A(B)=\tilde{\eta}(B)+(10/9)[(2C_1-A_1)\varepsilon_\mu]^{(C)}-[A_1\,\eta_T]^{(C)}, \tag{C11}$$

$$\eta_B(B)=\tilde{\eta}(B)+(5/18)[(8C_1+C_2+10C_3-4A_1-4A_2)\varepsilon_\mu]^{(C)}-[(A_1+A_2)\,\eta_T]^{(C)}+(\sqrt{2}\beta/24)\Psi(\sqrt{2}\beta), \tag{C12}$$

$$V_A(B)=\{(5/18)[(4A_2-C_2-10C_3)\varepsilon_\mu]^{(C)}+[A_2\,\eta_T]^{(C)}-(\sqrt{2}\beta/24)\Psi(\sqrt{2}\beta)\}(\ln|B|)', \tag{C13}$$

where $\Psi(x)=12[A_1'(x)+(1/2)A_2'(x)]\,\eta_T^{(M)}+\Psi_0'(x)\varepsilon_\mu^{(M)}$, $\tilde{\eta}(B)=[A_1(\beta)+(1/2)A_2(\beta)]\,\eta_T^{(P)}+(\varepsilon_\mu^{(P)}/12)\Psi_0(\beta)$, $\Psi_0(\beta)=(5/3)[4A_1(\beta)+3A_2(\beta)+4C_1(\beta)-C_3(\beta)]$ and $[X]^{(C)}$ is defined by Eq. (A25). The asymptotic formulas for the functions η_A, η_B, V_A, and $\alpha_{ij}^{(v)}$ for $\beta\ll1$ are given by

$$\eta_A(B)=\eta_*-(2/5)\beta^2[3\,\eta_T^{(v)}+(10/63)(14\varepsilon_\mu^{(v)}-\varepsilon_\mu^{(h)})], \tag{C14}$$

$$\eta_B(B)=\eta_*-(2/5)\beta^2[9\,\eta_T^{(v)}-8\,\eta_T^{(h)}+(10/63)(41\varepsilon_\mu^{(v)}-37\varepsilon_\mu^{(h)})], \tag{C15}$$

$$V_A(B)=(4/5)\beta^2[3\,\eta_T^{(v)}-4\,\eta_T^{(h)}+(5/7)(3\varepsilon_\mu^{(v)}-4\varepsilon_\mu^{(h)})]\times(\ln|B|)', \tag{C16}$$

$$\alpha_{ij}^{(v)}(\mathbf{B})=\delta_{ij}[(\alpha_0^{(v)}-(1/3)\varepsilon_\alpha)(1-(6/5)\beta^2)-(2/105)\beta^2\varepsilon_\alpha\epsilon], \tag{C17}$$

and for $\beta\gg1$ they are given by

$$\eta_A(B)=(\pi/6\beta)[(9/5)(\sqrt{2}-1)\eta_T^{(M)}+(\sqrt{2}-7/8)\varepsilon_\mu^{(v)} -(\sqrt{2}-9/8)\varepsilon_\mu^{(h)}], \tag{C18}$$

$$\eta_B(B)=(\pi/4\sqrt{2}\beta)\{(3/5)[(2\sqrt{2}-1)\eta_T^{(v)}+(2\sqrt{2}+1)\eta_T^{(h)}]+(1/24)[(22\sqrt{2}-13)\varepsilon_\mu^{(v)}+(18\sqrt{2}+13)\varepsilon_\mu^{(h)}]\}, \tag{C19}$$

$$V_A(B)=-\frac{3\pi}{4\sqrt{2}\beta}\left[\left(4\sqrt{\frac{2}{5}}-1\right)\{\eta_T^{(v)}+(5/8)\varepsilon_\mu^{(v)}\}+\eta_T^{(h)} +(5/8)\varepsilon_\mu^{(h)}\right](\ln|B|)', \tag{C20}$$

$$\alpha_{ij}^{(v)}(\mathbf{B})=-\delta_{ij}\{(\pi/\beta)\varepsilon_\alpha\epsilon-(2/\beta^2)[\alpha_0^{(v)}-(1/3)\varepsilon_\alpha(1-\epsilon)]\}. \tag{C21}$$

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